

LECTURE NOTES IN LOGIC

# SET THEORY, ARITHMETIC, AND FOUNDATIONS OF MATHEMATICS

THEOREMS, PHILOSOPHIES

EDITED BY  
JULIETTE KENNEDY  
ROMAN KOSSAK



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## **Set Theory, Arithmetic, and Foundations of Mathematics: Theorems, Philosophies**

This collection of papers from various areas of mathematical logic showcases the remarkable breadth and richness of the field. Leading authors reveal how contemporary technical results touch upon foundational questions about the nature of mathematics. Highlights of the volume include: a history of Tennenbaum's theorem in arithmetic; a number of papers on Tennenbaum phenomena in weak arithmetics as well as on other aspects of arithmetics, such as interpretability; the transcript of Gödel's previously unpublished 1972–1975 conversations with Sue Toledo, along with an appreciation of the same by Curtis Franks; Hugh Woodin's paper arguing against the generic multiverse view; Anne Troelstra's history of intuitionism through 1991; and Aki Kanamori's history of the Suslin problem in set theory.

The book provides a historical and philosophical treatment of particular theorems in arithmetic and set theory, and is ideal for researchers and graduate students in mathematical logic and philosophy of mathematics.

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## LECTURE NOTES IN LOGIC

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***Set Theory, Arithmetic,  
and Foundations of Mathematics:  
Theorems, Philosophies***

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*Edited by*

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ASSOCIATION FOR SYMBOLIC LOGIC



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Afraid! Of whom am I afraid?  
Not Death – for who is He?  
The Porter of my Father's Lodge  
As much abasheth me!

Of Life? 'Twere odd I fear [a] thing  
That comprehendeth me  
In one or two existences –  
As Deity decree –

Of Resurrection? Is the East  
Afraid to trust the Morn  
With her fastidious forehead?  
As soon impeach my Crown!

—Emily Dickinson



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# Introduction

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**§1. Introduction.** It is a unique feature of the field of mathematical logic, that almost any technical result from its various subfields: set theory, models of arithmetic, intuitionism and ultrafinitism, to name just a few of these, touches upon deep foundational and philosophical issues. What is the nature of the infinite? What is the significance of set-theoretic independence, and can it ever be eliminated? Is the continuum hypothesis a meaningful question? What is the real reason behind the existence of non-standard models of arithmetic, and do these models reflect our numerical intuitions? Do our numerical intuitions extend beyond the finite at all? Is classical logic the right foundation for contemporary mathematics, or should our mathematics be built on constructive systems? Proofs must be correct, but they must also be explanatory. How does the aesthetic of simplicity play a role in these two ideals of proof, and is there ever a “simplest” proof of a given theorem?

The papers collected here engage each of these questions through the veil of particular technical results. For example, the new proof of the irrationality of the square root of two, given by Stanley Tennenbaum in the 1960s and included here, brings into relief questions about the role simplicity plays in our grasp of mathematical proofs. In 1900 Hilbert asked a question which was not given at the Paris conference but which has been recently found in his notes for the list: find a criterion of simplicity in mathematics.<sup>1</sup> The Tennenbaum proof is a particularly striking example of the phenomenon Hilbert contemplated in his 24th Problem.

This collection of papers aims to review various points along the constructive/classical and finite/infinite divides, and accordingly, we feature here papers on ultrafinitism, intuitionism, models of arithmetic and set theory. On the side of foundations directly we include one of the last interviews with Kurt Gödel as

<sup>1</sup> See “Hilbert’s 24th Problem” by Rudiger Thiele.

well as a commentary on the same by Curtis Franks. The perspective of many of the papers is historical, reflecting this shift of emphasis in contemporary foundations of mathematics.

The inspiration behind this volume is the work and interests of the logician Stanley Tennenbaum, and through Tennenbaum the work of Kurt Gödel—a fundamental figure for Tennenbaum. Both saw mathematical logic as a unity, and both believed in philosophical analysis of a special kind: rationally optimistic in nature—not on the grounds of any genuinely abstract principle perhaps, but with a particular view of the nature of human understanding at its core.

**§2. Contents.** In 1959 Tennenbaum published an abstract announcing a theorem that became a standard introductory result in every presentation of model theory of nonstandard arithmetic structures: There are no computable presentations of nonstandard models Peano Arithmetic ( $\text{PA}$ ). On one hand, the result revealed severe limitations on possible constructions of nonstandard models; on the other hand, it opened a whole area of study around the problem: Which formal systems are subject to the Tennenbaum phenomenon? In other words, can we fully characterize formal systems which do not admit computable nonstandard models? Four papers presented in this volume give an account of how much we have learned about this problem in the last 50 years.

Richard Kaye looks at the history and discusses the origins Tennenbaum's theorem and earlier results of Kreisel, Mostowski, and Putnam, and shows a connection of to the Gödel-Rosser Incompleteness theorem: There is no consistent extension of  $\text{PA}$  whose restriction to  $\Pi_1$  sentences is computable. He then presents his own results on the Tennenbaum phenomenon for weak fragments of arithmetic. In particular, he proves that if  $T$  is an extension of  $\text{PA}$  in which the MRDP theorem is provable, then  $T$  has no computable nonstandard models.

Tennenbaum's proof shows that if  $(M, +, \times)$  is a nonstandard model of  $\text{PA}$ , then in fact neither  $(M, +)$  nor  $(M, \times)$  admit computable presentations. It follows that many other natural reducts of the structure  $M$  expanded by adding sets definable using  $+$  and  $\times$ , do not admit computable presentations. Schmerl's 1998 proof that some rich (in the non-technical sense) reducts can be effectively presented was therefore somewhat surprising. Schmerl's contribution to this volume provides a detailed analysis of the Tennenbaum phenomenon for reducts. Schmerl defines the notion of a generalized reduct and then considers the two cases: rich reducts and their refinements for which Tennenbaum Theorem holds. He then analyzes the much less studied case of thin and  $n$ -thin reducts, for which the theorem fails.

The whole area of study would not be that attractive without the other, algebraic, side. In 1964, Shepherdson gave an algebraic characterization of nonstandard models of Open Induction (OI) and used it to construct a computable nonstandard model of this theory. Shepherdson's model exhibits some pathological features. It contains a solution of the equation  $2x^2 = y^2$ . Efforts have been made to see how much of the theory of the standard model can be preserved in extensions of OI which escape the Tennenbaum phenomenon. The contributions of Mohsenipour and Raffer give all the necessary background to this area, and present some new results. Raffer is looking at nonstandard discretely ordered rings which satisfy OI and are diophantine correct, i.e. they satisfy all universal sentences true in the standard integers. DOI is the first order theory of such rings. It is not known if DOI admits nonstandard computable models. Raffer gives numbertheoretic conditions for a finitely generated ring of Puiseux polynomials to be diophantine correct. He uses a result on generalized polynomials to show that a certain class of ordered rings of Puiseux polynomials contains a model OI iff it contains a model of DOI.

$\text{OI}_n$  is a version of OI in which the induction schema is restricted to open formulas in which polynomial terms are of degree at most  $n$ . Boughattas proved that the theories  $\text{OI}_n$  form a hierarchy. Mohsenipour proves analogous results for theories extending OI. As a corollary he shows that none of the theories of the following classes of structures are finitely axiomatizable:  $\mathbb{Z}$ -rings, normal  $\mathbb{Z}$ -rings,  $\mathbb{Z}$ -rings satisfying the GCD axiom,  $\mathbb{Z}$ -rings with the Bezout property, and some relativized variants.

It is "common knowledge" that PA is essentially the same as  $\text{ZF}_{fin}$  which is ZF with the axiom of infinity replaced by its negation. If  $M$  is a model of PA, then bounded definable subsets of  $M$  form a model of  $\text{ZF}_{fin}$ , and if  $V$  is a model of  $\text{ZF}_{fin}$ , then the ordinals of  $V$  form a model of PA. However, Enayat, Schmerl, and Visser prove in their contribution to this volume that PA and  $\text{ZF}_{fin}$  are not bi-interpretable. In the process of proving it, the authors develop a new method for constructing models of  $\text{ZF}_{fin}$  and prove several interesting model-theoretic results, for example: every group can be realized as the automorphism group of a model of  $\text{ZF}_{fin}$ . They also prove a variant of Tennenbaum's theorem. While there are nonstandard (i.e. not isomorphic to the set of hereditarily finite sets  $V_\omega$ ) computable models of  $\text{ZF}_{fin}$ , every such model must be an  $\omega$ -model (i.e. every set in it has only a (standard) finite number of elements).

In set theory, Tennenbaum's work was centered on the Suslin Hypothesis (SH), to which, to quote from Aki Kanamori's contribution to this volume entitled "Tennenbaum and Set Theory," Tennenbaum made crucial contributions. These were to prove the relative consistency of its failure by forcing to

add a Suslin tree (1963), and with Solovay to prove its relative consistency, a result published only in 1971. Kanamori carefully parses the contribution of the two collaborators to the latter result. He also credits Tennenbaum with the asking of the question, at a time when Cohen forcing was more focused on cardinal arithmetic, collapsing cardinals and the like, than on classical questions in descriptive set theory.

In a different development in set theory, recent debate in its foundations has focused on the so-called “generic-multiverse position.” The generic-multiverse generated by a countable transitive model  $M$  of ZFC is defined to be the smallest set of countable transitive sets containing  $M$  and closed under the operations of generic extension and “generic submodel”;<sup>2</sup> the generic-multiverse is the generic-multiverse generated by  $V$ , and the generic-multiverse position holds that a sentence in the language of set theory is true if it is true in each universe of the generic-multiverse. Under the generic-multiverse position the Continuum Hypothesis is neither true nor false, as it holds in some universes of the generic-multiverse, but fails in others. On the other hand the status of Projective Uniformization, which is also independent of ZFC, is fundamentally different from that of the continuum problem. This is because Projective Uniformization<sup>3</sup> is either true in every universe of the generic-multiverse, or it is true in none of them, granting large cardinals.

The other set theory paper included in this volume, Hugh Woodin’s philosophical “The Continuum Hypothesis, the generic-multiverse of sets, and the  $\Omega$  Conjecture,” gives an accessible argument against the generic-multiverse position on the basis of the  $\Omega$  Conjecture and the existence of large cardinals. Woodin also argues against formalism in general as well as for the meaningfulness of the continuum problem, based on the (rather Gödelian) intuition that:

It seems incoherent to me to have a conception of the transfinite which reduces to simply a conception of  $H(\delta_0^+)$  which in essence is just the truncation of the universe of sets to the level of the least Woodin cardinal.

The paper formulates this intuition in exact terms, goes on to argue for it, and then poses an interesting and clear challenge to those who take a skeptical view of the meaningfulness of the continuum problem: exhibit a  $\Pi_2$  assertion which is true and which is not true across the generic-multiverse.

<sup>2</sup> I.e. if  $M$  belongs to the generic-multiverse and generically extends a countable transitive model  $N$  of ZFC, then  $N$  belongs to the generic-multiverse.

<sup>3</sup> I.e. the statement that for every projective subset of the Cartesian plane there is a projective choice function.

“A Very Short History of Ultrafinitism,” by Rose Cherubin and Mirco Mannucci, is an ultrafinitist manifesto which first traces the various uses of the terms “finite” and “infinite” in Homer, respectively *murios* and *apeiron*, as well as in some subsequent classical sources, and then considers the contemporary history of Ultrafinitism in the light of this background. In particular they argue for a context-based, dynamic notion of feasibility. Cherubin and Mannucci are<sup>4</sup> former students of Stanley Tennenbaum (in the informal sense of the term of course), so it is not surprising that their contribution reflects Tennenbaum’s approach to logic, if not his general intellectual spirit, so faithfully.

Also on the philosophical side we include the notes of the proof theorist Sue Toledo’s conversations with Kurt Gödel about phenomenology, proof theory, intuitionism, finitism, and the Euthyphro, conversations which took place at Tennenbaum’s instigation. They span the period from 1972–1975, making these just about the last words we have from Gödel. We publish them here without commentary, as that is what Tennenbaum would have wished. The notes, though fragmentary in a few places, add considerably to the general picture we have of the late period of Gödel’s life and thought. We are very grateful to Sue Toledo for giving us permission to publish them.

Curtis Franks’s “Stanley Tennenbaum’s Socrates,” a penetrating and beautiful tribute to these conversations, explores the parallel Plato/Phaedo/Socrates versus Tennenbaum/Toledo/Gödel. One obvious connection is that the last sustained thoughts of Socrates, as presented by Plato, are to be found in the Phaedo; and similarly we find in the Toledo conversations some of the last thoughts of Gödel, as presented in some weaker sense by Tennenbaum—where both presentations are given in the form of conversations. A more subtle connection is that Socrates is reflecting on his “career” as a philosopher, in the Phaedo, in a way strikingly similar to Gödel’s reflection on his own intellectual career here. Franks’s reading of Gödel reveals a unity in the various elements of Gödel’s thought, which verges on the sublime. But the device of the piece yields still more: namely, the unmistakable voice of Stanley Tennenbaum, whose world view comes through here very clearly.

We are privileged to include Anne Troelstra’s history of intuitionism through 1991, reflecting Tennenbaum’s long interest in the work of Errett Bishop.

Finally we include Tennenbaum’s entirely geometric proof of the irrationality of the square root of 2. It appears to be a genuinely new proof of that theorem, as well as the simplest ever found.

*The Editors*

Juliette Kennedy

Roman Kossak

<sup>4</sup> along with Kennedy and Raffer.



# HISTORICAL REMARKS ON SUSLIN'S PROBLEM

AKIHIRO KANAMORI

The work of Stanley Tennenbaum in set theory was centered on the investigation of Suslin's Hypothesis (SH), to which he made crucial contributions. In 1963 Tennenbaum established the relative consistency of  $\neg$ SH, and in 1965, together with Robert Solovay, the relative consistency of SH. In the formative period after Cohen's 1963 discovery of forcing when set theory was transmuting into a modern, sophisticated field of mathematics, this work on SH exhibited the power of forcing for elucidating a classical problem of mathematics and stimulated the development of new methods and areas of investigation.

§ 1 discusses the historical underpinnings of SH. § 2 then describes Tennenbaum's consistency result for  $\neg$ SH and related subsequent work. § 3 then turns to Tennenbaum's work with Solovay on SH and the succeeding work on iterated forcing and Martin's Axiom. To cast an amusing sidelight on the life and the times, I relate the following reminiscence of Gerald Sacks from this period, no doubt apocryphal: Tennenbaum let it be known that he had come into a great deal of money, \$30,000,000 it was said, and started to borrow money against it. Gerald convinced himself that Tennenbaum seriously believed this, but nonetheless asked Simon Kochen about it. Kochen replied, "Well, with Stan he might be one per-cent right. But then, that's still \$300,000."

**§1. Suslin's problem.** At the end of the first volume of *Fundamenta Mathematicae* there appeared a list of problems with one attributed to Mikhail Suslin [1920], a problem that would come to be known as Suslin's Problem. After the reunification of Poland in 1918, there was a deliberate decision by its aspiring mathematicians to focus on set theory and related areas and to bring out a new journal to promote international research<sup>1</sup>. This was the origin of *Fundamenta Mathematicae*, which became the main conduit of scholarship in "fundamental mathematics" during the 1920s and 1930s. That first list of problems had to do with possible consequences of the Continuum Hypothesis

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This is an expanded version of an invited address given at a memorial conference commemorating the life and work of Stanley Tennenbaum held at the Graduate Center of the City University of New York on 7 April 2006.

<sup>1</sup>See Kuratowski [1980] and Kuzawa [1968].

(CH) or issues in the emerging descriptive set theory. These problems would be solved, but by contrast Suslin's Problem would grow in significance through its irresolution.

Georg Cantor, the founder of set theory, had famously characterized the ordertypes of the rationals and reals in the second part [1897] of his *Beiträge*, his mature presentation of his theory of the transfinite. The ordering of the reals is that unique dense linear ordering with no endpoints which is order-complete (i.e. every bounded set has a least upper bound<sup>2</sup>) and separable (i.e. has a countable dense subset). *Suslin's Problem* asks whether this last condition can be weakened to the countable chain condition (c.c.c.): every disjoint family of open intervals is countable. Although Suslin himself did not hypothesize it, the affirmative answer has come to be known as *Suslin's Hypothesis* (SH). For a dense linear order, deleting endpoints and taking the (Dedekind) completion does not affect the c.c.c. or separability properties. So, the hypothesis could be simply stated as:

(SH) Every (infinite) dense linear ordering satisfying the c.c.c. is separable.

Suslin's Problem would be the first anticipation of the study of chain conditions in general topology, and as such it displays a remarkable foresight. Suslin himself was a wunderkind who after finding a mistake in a paper of Lebesgue formulated the analytic sets and established [1917] fundamental results about them: a set of reals is Borel exactly when it and its complement are analytic, and there is an analytic set which is not Borel. These seminal results considerably stimulated the Soviet and Polish schools in descriptive set theory, and some of the problems on that first *Fundamenta* list concerns analytic sets<sup>3</sup>. [1917] was to be Suslin's sole publication<sup>4</sup>, for he succumbed to typhus in the 1919 Moscow epidemic at the age of 25. Until the early 1970s one sees the "Souslin" from Suslin [1920]; this is the French transliteration, for *Fundamenta* at first adopted French as the *lingua franca*.

Recapitulating the mathematical experience, it is hard to see how to go from the rather amorphous countable chain condition to a countable dense subset. To compare, Cantor had first formulated CH as the loose assertion that there is no strictly intermediate power between that of the natural numbers and that of the continuum. Cantor then developed the transfinite numbers and converted CH to the positive, existence assertion that there *is* a bijection between the continuum and the countable ordinals, and thereafter tried to exploit analogies between increasing convergent sequences of reals and such

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<sup>2</sup>Cantor had every fundamental (Cauchy) sequence being complete, but this has to be mediated by the Axiom of Choice.

<sup>3</sup>See Kanamori [1995] for the emergence of descriptive set theory.

<sup>4</sup>Suslin [1920] is an attribution to Suslin. It should be observed that accompanying Suslin [1917] was a note by Nikolai Luzin [1917] in which he credited Suslin with having established that every analytic set has the perfect set property.

sequences of countable ordinals. With Suslin's Problem, in the several decades after its articulation what interest there was in it became focused on  $\neg\text{SH}$ , a positive, existence assertion which became characterized in a perspicuous form that suggested possibilities for establishing it. Both mathematically and historically, when "Suslin's Hypothesis" came into use it should thus have arguably referred to  $\neg\text{SH}$ .

A *tree* is a partially ordered set  $\langle T, <_T \rangle$  with a minimum element such that for any  $x \in T$  the set  $\{y \in T \mid y <_T x\}$  of its  $<_T$ -predecessors is well-ordered by  $<_T$ . The  $\alpha$ th level of  $T$  consists of those  $x \in T$  whose set of  $<_T$ -predecessors has order type  $\alpha$  under  $<_T$ . The *height* of  $T$  is the least  $\alpha$  such that the  $\alpha$ th level of  $T$  is empty. A *chain* of  $T$  is a linearly ordered subset, and an *antichain* of  $T$  is a subset consisting of pairwise  $<_T$ -incomparable elements. A *Suslin tree* is a tree of height  $\omega_1$  with no uncountable chains or antichains.

Trees abound in contemporary set theory as basic combinatorial objects<sup>5</sup>. The first systematic study of trees was carried out in Kurepa's dissertation [1935] with Fréchet, where several tree and linear order equivalences were derived. Kurepa [1936, 127ff] provided the following characterization, since rediscovered by Edwin Miller [1943] and Waclaw Sierpiński [1948]:

$\neg\text{SH}$  *iff* there is a Suslin tree.

In the forward direction, let  $\langle S, <_S \rangle$  be a counterexample to SH, i.e. a dense linear ordering with the c.c.c. but with no countable dense subset. Recursively construct non-empty open intervals  $I_\alpha$  for  $\alpha < \omega_1$  as follows: Let  $I_0$  be  $S$ . Given  $I_\beta$  for  $\beta < \alpha$ , since the set  $E$  of all the endpoints of these intervals is countable, let  $I_\alpha$  be an interval disjoint from  $E$ . Then  $\{I_\alpha \mid \alpha < \omega_1\}$  under reverse inclusion is a Suslin tree.

In the converse direction, let  $\langle T, <_T \rangle$  be a Suslin tree. By successively pruning and omitting intermediate nodes, we can assume that: every element has uncountably many successors; different elements at a limit level do not have the same sets of predecessors; and every element has more than one immediate successor. We can further assume, by restricting to the limit levels only, that every element has infinitely many immediate successors. Now, linearly order each level as a dense linear order without endpoints. Then let  $S$  consist of the maximal chains ("branches") of  $T$ , and for  $c_1 \neq c_2 \in S$ , define  $c_1 <_S c_2$  exactly when at the least level at which they differ, the element in  $c_1$  precedes the element in  $c_2$  in that level's linear order. Then  $\langle S, <_S \rangle$  is a counterexample to SH.

This tree characterization of  $\neg\text{SH}$  eliminated topological considerations from Suslin's Problem and reduced it to a problem of combinatorial set theory. Suslin trees and their generalizations have since played important roles in modern set theory both in providing examples and in promoting the development of set-theoretic methods. Even early on,  $\neg\text{SH}$  led to examples in

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<sup>5</sup>See Todorčević [1984] for a magisterial account.

general topology (cf. Rudin [1955]) and played a role analogous to CH — an unproved hypothesis from which significant conclusions were drawn. This incidentally, is further reason why “Suslin’s Hypothesis” should be attached to  $\neg$ SH. In any case,  $\neg$ SH, like CH, would have to await Cohen’s discovery of forcing for significant elucidation.

**§2. Consistency of  $\neg$ SH.** Making the first real breakthrough on Suslin’s Problem since its articulation, Tennenbaum [1968] in 1963 established the relative consistency  $\text{Con}(\text{ZFC})$  implies  $\text{Con}(\text{ZFC} + \neg\text{SH})$ . This he did by devising a notion of forcing for adding a Suslin tree. Proceeding in modern vein, Tennenbaum’s partial order consists of finite trees  $\langle t, <_t \rangle$ , where  $t \subseteq \omega_1$  and  $\alpha <_t \beta$  implies  $\alpha < \beta$ , ordered by:

$$\langle t_1, <_{t_1} \rangle \leq \langle t_2, <_{t_2} \rangle \text{ iff } t_1 \supseteq t_2 \wedge <_{t_2} = <_{t_1} \cap (t_2 \times t_2).$$

That this partial order has the countable chain condition<sup>6</sup> can be established by a typical  $\Delta$ -system argument, and hence forcing with it preserves all cardinals.

For a generic  $G$ , define

$$T = \bigcup \{t \mid \langle t, <_t \rangle \in G\}, \quad \text{and} \quad < = \bigcup \{<_t \mid \langle t, <_t \rangle \in G\},$$

$\langle T, < \rangle$  is a tree, of height  $\omega_1$  by a density argument. Another  $\Delta$ -system argument establishes that this tree has no uncountable antichains. Finally, this tree cannot have any uncountable chains either, as such a chain would engender an uncountable antichain consisting of “offshoots”. Hence,  $\langle T, < \rangle$  is a Suslin tree.

To the set theorist of today this consistency result is quite straightforward, and even Tennenbaum once told me that it was merely an “exercise” in forcing. However, it is remarkable that, according to a footnote of Tennenbaum [1968], the work was done in the summer of 1963. Cohen had just come up with forcing that spring and had established his relative consistency results about the Continuum Hypothesis and the Axiom of Choice. With Solomon Feferman, Robert Solovay, and Azriel Levy, Tennenbaum would be among the first after Cohen who established results with forcing. Moreover, Tennenbaum’s notion of forcing was the first to address issues in ZFC other than the sort that Cohen himself addressed, about powers of cardinals, collapsing cardinals, and definability. Tennenbaum expressed gratitude to Georg Kreisel, Anil Nerode, and Dana Scott for pointing out gaps in previous attempts and to Kurt Gödel, who communicated the paper, for simplifying the countable chain condition argument.

Several years later Thomas Jech [1967] independently established the consistency of  $\neg$ SH. Working in Prague, he approached the result through Petr

<sup>6</sup>This of course is in the well-known sense for a forcing partial order. The c.c.c. defined earlier will be consistent with this, if we take the open sets of the topology under the partial order of inclusion.

Vopěnka's  $\nabla$ -models. Of note, there was earlier Russian–Eastern European work by Yesenin-Volpin [1954], who had produced a Fraenkel-Mostowski permutation model of  $\neg\text{SH}$  with continuum many urelements. Jech's forcing conditions, unlike Tennenbaum's, are countable approximations to a Suslin tree ordered by end-extension. Cardinals are preserved because of the countable closure of the partial order, and that the resulting tree has no uncountable antichains is established by a Skolem hull argument. On the one hand, this argument is more involved than its counterpart in Tennenbaum's proof, but on the other hand, it is Jech's proof that would generalize in the later work on higher cardinality versions of Suslin trees. From the modern perspective it is more natural to consider initial segments of a possible Suslin tree as conditions, but Tennenbaum presumably worked out his approach because of its affinity to Cohen's for violating CH; countable closure of forcing conditions was not used by Cohen, but he evidently used the countable chain condition. Also, Tennenbaum's argument works whether CH or  $\neg\text{CH}$  holds and preserves that state of affairs; Jech's forcing always enforces CH.

The most significant result about  $\neg\text{SH}$ , one that moreover would have the most potentiality, was the result of Ronald Jensen [1968] that for Gödel's constructible universe  $L$ , *if  $V = L$ , then there is a Suslin tree*. Jensen famously isolated a combinatorial principle  $\diamond$  that carried what was needed of the structure of  $L$  and showed that  $\diamond$  itself implies that there is a Suslin tree. This would not only lead to generalizations to higher cardinals but also begin Jensen's broad investigation of pivotal combinatorial principles holding in  $L$ , principles that would achieve autonomous status for establishing a wide range of propositions of combinatorial set theory.

There is an underlying similarity between Tennenbaum's argument and Jensen's for establishing that the tree has no uncountable antichains. At base there is a common forcing argument for how levels of the tree are to be generically extended. Jech's forcing is actually equivalent to the forcing  $Q$  for adding a Cohen subset of  $\omega_1$  with countable conditions<sup>7</sup>.  $Q$  actually adjoins  $\diamond$ , and hence by Jensen's result that there is a Suslin tree.

Tennenbaum's forcing is rather like the standard forcing for adding  $\aleph_1$  reals (with finite conditions). Saharon Shelah [1984] eventually established that adding even one Cohen real adjoins a Suslin tree. This last is a remarkable result, and provides a structured sense to the assertion that one is always very close to having a Suslin tree.

**§3. Consistency of SH.** Making those moves in set theory that would be the most consequential not only for Suslin's Problem but for the development of forcing in general, Tennenbaum began to investigate the possibility of

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<sup>7</sup>And  $Q$  is in turn equivalent to the usual forcing for collapsing  $2^{\aleph_0}$  to  $\omega_1$ , since every condition in  $Q$  has  $2^{\aleph_0}$  many incompatible extensions.

establishing the consistency of SH. It had been simple enough to add a Suslin tree by forcing, and now there was the difficult prospect of getting a model with *no* Suslin trees. Tennenbaum saw how “to kill a Suslin tree”: That a Suslin tree has no uncountable antichains is exactly the property of the partial order having the countable chain condition, and so forcing with the tree itself generically adds an uncountable chain. In fact, it is easy to see that any uncountable chain actually corresponds to a generic object, and so genericity was intrinsic to having an uncountable chain. Here for the first time was the recasting of a combinatorial object of classical mathematics as itself a notion of forcing. But what now? Once a Suslin tree is thus killed it stays dead, but other Suslin trees might have sprung up, and so they too must be killed by iterating the process. This is where the collaboration with Robert Solovay came in.

Robert Solovay above all epitomized this period of great expansion in set theory, with his mathematical sophistication and his fundamental results with forcing, in large cardinals, and in descriptive set theory. Following initial graduate study in differential topology, Solovay focused his energies on set theory after attending a lecture of Cohen’s in May 1963. Solovay first extended the independence of CH by characterizing the possibilities for the size of the continuum, and then generalized Cohen’s forcing to arbitrary partial orders and dense sets. He next established his famous Lebesgue measurability result during March-July 1964 (Solovay [1970, 1]). Then with Tennenbaum he worked out the iterated forcing proof of the consistency of SH (Solovay and Tennenbaum [1971, 201]). An email letter of 18 February 2006 from Solovay to the author describes the interaction with Tennenbaum and the main features of their proof for  $\text{Con}(\text{ZFC})$  implies  $\text{Con}(\text{ZFC} + \text{SH})$ ; the letter is quoted here with Solovay’s permission, verbatim except for a side remark at the first (b).

Aki,

Sorry to have taken so long to get back to your request re Tennenbaum. I hope these comments are not too late.

At one crucial point in the following my memory is not clear and I have tried to reconstruct what happened. I’ve labeled the reconstruction as speculation in what follows.

I was at the Institute for Advanced Study in Princeton during the year 1964-65. I think Stan was teaching somewhere in Philadelphia at the time. At any rate, he would come up regularly to Princeton to talk about set theory and more specifically his attempt to prove the consistency of Souslin’s Hypothesis.

At that time, he had already proved the independence of SH via a forcing argument where the conditions were finite approximations to the Souslin Tree that was to be generically added.

His attempt/plan for the consistency of SH had the following ingredients:

- (a) it was to be an iteration in which at each step another Souslin tree would be killed.
- (b) The steps in the iteration were to be forcing with Souslin trees. . . .
- (c) Stan knew that forcing with a Souslin tree killed it and that the forcing was c.c.c.

For much of the year, Stan was trying to prove that the iteration did not collapse cardinals. And he was considering iterations of lengths 2 and 3. My role was to passively listen to his proofs and spot the errors in them. With unjustified prescience, I kept saying that I was worried about “killing the same tree twice”. Of course, there now are examples due to Jensen that show that forcing with the product of a Souslin tree with itself can collapse cardinals. But these weren't available then.

At one of those meetings one of us (probably Stan) made progress and finally found reasonable conditions under which a two stage iteration did not collapse cardinals.

Somehow this got me seriously thinking about the problem and by the time of our next meeting I had a proof of the theorem. This proof had the following ingredients which were new:

- (a) defining a transfinite sequence of forcing notions  $P_\alpha$  where  $P_{\alpha+1} = P_\alpha * Q_\alpha$ .

This involves: (a1) defining the operation  $*$  where  $P * Q$  is defined if  $Q$  is a poset in  $V^P$ ;

- (a2) defining what to do at limits.
- (b) proving that if the component forcings are c.c.c. then the limit forcing is c.c.c.;
- (c) exploiting the c.c.c.ness to see that if the length of the iteration has cofinality greater than  $\omega_1$  then all subsets of  $\aleph_1$  in the final model appear at some proper intermediate stage;
- (d) setting up the bookkeeping so that all Souslin trees are killed.

I admit that, with hindsight, all these things look easy now.

I come now to the speculation as to what was discovered that day that set me to thinking. My guiding principle is that it has to be something which is not utterly trivial.

I think that Stan was looking at the product of two Souslin trees in the initial ground and trying to show the product was c.c.c.

Of course, with this level of generality this is not true. I suspect that the discovery was that if the second tree  $T_2$  remained Souslin in  $V[T_1]$  then the composed forcing was c.c.c.

When it came time to write the proof up, I was under the spell of Boolean valued models [which I had just discovered] and asked Stan if I could present

the proof in terms of them. He agreed, but subsequently had strong reservations, and the paper was almost not published. [He eventually agreed to its publication with a footnote expressing his reservations about the Boolean valued approach. The footnote is in the published paper.] Whether I would have had the stomach to withdraw the paper from the *Annals* [it had already been accepted] and rewrite it completely, I don't know. I'm glad it didn't come to that.

I hope these comments are useful to you. As I write them I am keenly aware of the fallibility of memory after all this time. To use the cliché, these are “my best recollections”.

—Bob

Speaking to the last substantive paragraph, there seem to be no reservations about the Boolean-valued approach expressed on behalf of Tennenbaum in Solovay and Tennenbaum [1971], in footnotes or elsewhere. I was bemused when once Tennenbaum told me that he did not understand their paper. The idea of assigning to a formula in the forcing language a value from a complete Boolean algebra had occurred to Solovay around the time of the collaboration with Tennenbaum; Solovay conveyed the idea to Scott, and by late 1965 they had independently come to the idea of starting with Boolean valued sets from the beginning to do forcing<sup>8</sup>. Boolean-valued models still held sway when Solovay wrote up Solovay and Tennenbaum [1971], which was submitted in late 1969.

The several years between proof and write-up allowed for a significant incorporation. Around 1967 Tony Martin observed that “by the same techniques used by the authors to get a model of SH” (Solovay and Tennenbaum [1971, 232]) one can establish the consistency of a stronger, focal “axiom”, and Solovay named this axiom and incorporated its consistency into the write-up. This was the genesis of Martin's Axiom (MA), independently suggested by Kenneth Kunen and Frederick Rowbottom. Beyond killing all the Suslin trees, one can analogously add “generics” to all countable chain condition forcings if one only has to meet less than continuum many dense sets. With its consistency in hand Martin and Solovay [1970] showed that MA has a wide range of consequences, as it allows for constructions analogous to those from CH. Since then of course, MA has achieved a methodological prominence as a focal “axiom” for relative consistency results.

Solovay and Tennenbaum [1971] is the beginning of genuine, iterated forcing in the following sense. Before their work, the major accomplishments with forcing that had involved the analysis of product forcings had been the work of William Easton [1964], [1970] on powers of regular cardinals and the work of Solovay [1965], [1970] on Lebesgue measurability. However, in these cases

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<sup>8</sup>See Bell [1985, xivff].

the products were taken of ground model notions of forcing. Solovay and Tennenbaum [1971] was the first to work with forcings analyzed as iterating one notion of forcing followed by a subsequent one which only occurs in an intermediate generic extension and to show that combinatorial properties like the countable chain condition are preserved. Iterated forcing is part and parcel of modern set theory, and it was born that day in 1965.

Considering the further development of set theory, the sophistication of the algebraic Boolean-valued approach in Solovay and Tennenbaum [1971] had the effect of obscuring the underlying iteration processes. Soon set theorists were back to working entirely with partial orders. William Mitchell [1973] specifically stated the product lemma for the two-step iteration of partial orders, and Richard Laver [1976] cast his iterated forcing even through limit stages in the modern way with sequences of conditions. Around this time I recall Jack Silver being enthused that the consistency of MA can be done with partial orders.

On a final note, the model of Solovay and Tennenbaum [1971] inherently satisfies  $\neg\text{CH}$ . In a *tour de force* of forcing for the time, Jensen (cf. Devlin and Johnsbråten [1974]) established the joint consistency  $\text{Con}(\text{ZFC})$  implies  $\text{Con}(\text{ZFC} + \text{CH} + \text{SH})$ . This he did by a forcing that adds no new reals, kills Souslin trees by “specializing” them, and is constructed with special inverse limits based on his combinatorial principles derived from  $L$ . Saharon Shelah once told me that it was trying to come to grips with Jensen’s argument that inspired a significant part of his early work with his “proper” forcing. Shelah [1998, V § 6] establishes Jensen’s result systematically in his proper forcing context.

In an address given at the International Congress of Logic, Methodology, and Philosophy of Science held at Bucharest in 1971, Alfred Tarski reserved particular praise for Solovay’s work on Lebesgue measurability and Solovay and Tennenbaum’s work on Suslin’s Hypothesis as the most important in set theory since Cohen’s work<sup>9</sup>. In a letter to Gödel of 6 April 1973, Abraham Robinson wrote: “As the outstanding individual achievements in logic since 1963 I would list (in chronological order): 1) The work of Ax-Kochen on  $p$ -fields. 2) Solovay on measurable sets, and the Tennenbaum-Solovay on Souslin’s conjecture. 3) Matiyasevich’s solution of Hilbert’s 10th problem.”

**§4. Envoi.** My own encounters with Tennenbaum were modest. When I was an instructor at UC Berkeley, my first job, Tennenbaum sauntered into my office one day in the spring of 1976, introduced himself, and promptly made himself comfortable by sitting on the floor. I of course was aware of his work on SH, and in my youthful reckoning of people by their recent accomplishments was somewhat taken aback by all this as well as his general disheveled

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<sup>9</sup>This is the recollection of the author, who as a research student attended the conference.

appearance despite sport-jacket and tie. Anyway, we had a pleasant conversation about my work and prospects in set theory, from which I mainly recall his steady probing for connections and meanings. At one point he asked me for my “pedigree” and I charted my mathematical genealogy on the blackboard: Kanamori — Mathias — Jensen — Hasenjaeger — Scholz. He had heard of Mathias and, of course, Jensen. I explained that Hasenjaeger was a German set theorist who had something to do with the Completeness Theorem, and that through him I was connected to Heinrich Scholz, a theologian who kept mathematical logic alive in Germany between the world wars. For some reason, to this day I can still visualize Tennenbaum’s eyes suddenly opening wide and his hands going up to his chin as he pondered the blackboard.

Years later I was touring Montreal with a young family, and I suddenly saw Tennenbaum across the street. It was something about his clothes or the angularity of his walk — he was unmistakable. I was busily shepherding children *en train* and decided that I could not possibly cross the street and go up to him. But peering out of the corner of my eye, I espied him calmly sauntering up the narrow street, looking this way and that, focusing for a moment on crocuses in a flowerpot. I had recently seen the film *Being There*, with Peter Sellers in the main role of Chauncy Gardner, and I was pointedly reminded of the final scene, where Gardner is seemingly walking on water, enjoying the water lilies.

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# THE CONTINUUM HYPOTHESIS, THE GENERIC-MULTIVERSE OF SETS, AND THE $\Omega$ CONJECTURE

W. HUGH WOODIN

**§1. A tale of two problems.** The formal independence of Cantor's Continuum Hypothesis from the axioms of Set Theory (ZFC) is an immediate corollary of the following two theorems where the statement of the Cohen's theorem is recast in the more modern formulation of the Boolean valued universe.

**THEOREM 1** (Gödel, [3]). *Assume  $V = L$ . Then the Continuum Hypothesis holds.*

**THEOREM 2** (Cohen, [1]). *There exists a complete Boolean algebra,  $\mathbb{B}$ , such that*

$$V^{\mathbb{B}} \models \text{“The Continuum Hypothesis is false”}.$$

Is this really evidence (as is often cited) that the Continuum Hypothesis has no answer?

Another prominent problem from the early 20th century concerns the *projective sets*, [8]; these are the subsets of  $\mathbb{R}^n$  which are generated from the closed sets in finitely many steps taking images by continuous functions,  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , and complements. A function,  $f : \mathbb{R} \rightarrow \mathbb{R}$ , is projective if the graph of  $f$  is a projective subset of  $\mathbb{R} \times \mathbb{R}$ . Let *Projective Uniformization* be the assertion:

*For each projective set  $A \subset \mathbb{R} \times \mathbb{R}$  there exists a projective function,  $f : \mathbb{R} \rightarrow \mathbb{R}$ , such that for all  $x \in \mathbb{R}$  if there exists  $y \in \mathbb{R}$  such that  $(x, y) \in A$  then  $(x, f(x)) \in A$ .*

The two theorems above concerning the Continuum Hypothesis have versions for Projective Uniformization. Curiously the Boolean algebra for Cohen's theorem is the same in both cases, but in case of the problem of Projective Uniformization an additional hypothesis on  $V$  is necessary. While Cohen did not explicitly note the failure of Projective Uniformization, it is arguably implicit in his results. On the other hand Gödel was aware and did note that Projective Uniformization does hold in  $L$ ; he calculated that if  $V = L$  then

there is a wellordering of the reals which as a binary relation is a projective subset of the plane.

THEOREM 3 (Gödel, [3]). *Assume  $V = L$ . Then Projective Uniformization holds.*

THEOREM 4 (Cohen, [1]). *Assume  $V = L$ . There exists a complete Boolean algebra,  $\mathbb{B}$ , such that*

$$V^{\mathbb{B}} \models \text{“Projective Uniformization does not hold”}.$$

Thus one also obtains the formal independence of Projective Uniformization from the axioms of Set Theory. But in this case this is *not* evidence that the problem of Projective Uniformization has no answer. The reason is that there is a fundamental difference in the problem of the Continuum Hypothesis versus the problem of Projective Uniformization—it was realized fairly soon after Cohen’s initial results that *large cardinal* axioms could not be used to settle the Continuum Hypothesis, [7].

THEOREM 5 (Levy, Solovay [7]). *There exists a complete Boolean algebra,  $\mathbb{B}$ , such that*

$$V^{\mathbb{B}} \models \text{“The Continuum Hypothesis”}.$$

Assuming the consistency of certain large cardinal axioms, the analogous theorem for negation of Projective Uniformization is *false*. The reason is that large cardinal axioms have been shown to imply that Projective Uniformization holds. There are several versions of this theorem and the most recent and essentially optimal version (optimal in its economy of large cardinal axioms) is given by the seminal theorem of Martin and Steel [9], rephrased here to be relevant to this discussion.

THEOREM 6 (Martin, Steel). *Assume there are infinitely many Woodin cardinals. Then Projective Uniformization holds.*

COROLLARY 7. *Suppose there is a proper class of Woodin cardinals. Then for all complete Boolean algebras,  $\mathbb{B}$ ,*

$$V^{\mathbb{B}} \models \text{“Projective Uniformization”}.$$

**§2. The generic-multiverse of sets.** Let the *multiverse* (of sets) refer to the collection of possible universes of sets. The truths of the Set Theory are the sentences which hold in each universe of the multiverse. The multiverse is the *generic-multiverse* if it is generated from each universe of the collection by closing under generic extensions (enlargements) and under generic refinements (inner models of a universe which the given universe is a generic extension of). To illustrate the concept of the generic-multiverse, suppose that  $M$  is a countable transitive set with the property that

$$M \models \text{ZFC}.$$

Let  $\mathbb{V}_M$  be the smallest set of countable transitive sets such that  $M \in \mathbb{V}_M$  and such that for all pairs,  $(M_1, M_2)$ , of countable transitive sets such that

$$M_1 \models \text{ZFC},$$

and such that  $M_2$  is a generic extension of  $M_1$ , if either  $M_1 \in \mathbb{V}_M$  or  $M_2 \in \mathbb{V}_M$  then both  $M_1$  and  $M_2$  are in  $\mathbb{V}_M$ . It is easily verified that for each  $N \in \mathbb{V}_M$ ,

$$\mathbb{V}_N = \mathbb{V}_M,$$

where  $\mathbb{V}_N$  is defined using  $N$  in place of  $M$ .  $\mathbb{V}_M$  is the generic-multiverse generated in  $V$  from  $M$ .

The *generic-multiverse position* is the position that a sentence is true if and only if it holds in each universe of the generic-multiverse generated by  $V$ . A priori this conception of truth seems to depend on (and therefore require) a larger universe within which the generic-multiverse generated by  $V$  is computed. However this conception of truth can actually be formalized within  $V$  without regard to any such larger universe. More precisely, for each sentence  $\phi$  there is a sentence  $\phi^*$ , recursively depending on  $\phi$ , such that  $\phi$  is true in each universe of the generic-multiverse generated by  $V$  if and only if  $\phi^*$  is true in  $V$ . The recursive transformation which sends  $\phi$  to  $\phi^*$  is explicit (we shall actually specify it) and does not depend on  $V$ . Thus with  $M$  and  $\mathbb{V}_M$  as above, the following are equivalent.

- (1)  $M \models \phi^*$ .
- (2)  $N \models \phi$  for all  $N \in \mathbb{V}_M$ .
- (3)  $N \models \phi^*$  for all  $N \in \mathbb{V}_M$ .

Since this is an important point in favor of the generic-multiverse position, I give a proof in the appendix. The relevance of this point to the generic-multiverse position is that it shows that as far as assessing truth, the generic-multiverse position is not that sensitive to the meta-universe in which the generic-multiverse is being defined.

The generic-multiverse position has a feature which the multiverse view given by formalism does *not* share: the notion of truth is the same as defined relative to each universe of the multiverse, so from the perspective of evaluating truth *all* the universes of the multiverse are equivalent. This seems another important point in favor of the generic-multiverse position and this point is reinforced by the reduction indicated above of truth in the generic-multiverse to truth in each constituent universe.

In fact the multiverse position given by essentially any generalization of first order logic is *not* the same as defined relative to each universe of the multiverse (the definition of the logic is not absolute to each universe of the multiverse). This only requires that the logic be definable and a more precise version of this claim is given in the following lemma the statement of which requires some notation.

Suppose  $\Phi(x)$  is a formula and define  $T_\Phi$  to be the set of all sentences  $\psi$  such that if  $\mathcal{M} \models \text{ZFC}$  and if  $\Phi[\mathcal{M}]$  holds then  $\mathcal{M} \models \psi$ . Thus if there is no such model  $\mathcal{M}$ ,  $T_\Phi$  is simply the set of all sentences. For each model,  $\mathcal{M} \models \text{ZFC}$ , let  $(T_\Phi)^\mathcal{M}$  be the set of all sentences  $\phi$  such that

$$\mathcal{M} \models "\phi \in T_\Phi."$$

Let  $\hat{\Phi}$  be a Gödel sentence,  $\phi$ , which expresses:  $\phi \notin T_\Phi$ . More precisely, let  $\hat{\Phi}$  be a sentence such that for all models  $\mathcal{M} \models \text{ZFC}$ ,  $\mathcal{M} \models \hat{\Phi}$  if and only if  $\hat{\Phi} \notin (T_\Phi)^\mathcal{M}$ .

The following lemma generalizes Gödel's Second Incompleteness Theorem.

LEMMA 8. *Suppose that there exists a model,  $\mathcal{M} \models \text{ZFC}$ , such that  $\Phi[\mathcal{M}]$  holds. Then there exists a model,  $\mathcal{M} \models \text{ZFC}$ , such that  $\Phi[\mathcal{M}]$  holds and such that either:*

- (1)  $\hat{\Phi} \in T_\Phi$  and  $\hat{\Phi} \notin (T_\Phi)^\mathcal{M}$ , or
- (2)  $\hat{\Phi} \notin T_\Phi$  and  $\hat{\Phi} \in (T_\Phi)^\mathcal{M}$ .

The proof of the lemma is immediate from the definitions. In general the conclusion of the lemma is best possible. More precisely, assuming there is a model of ZFC there are examples of  $\Phi$  for which (1) does not hold with  $\hat{\Phi}$  replaced by any sentence whatsoever and assuming there is an  $\omega$ -model of ZFC (less suffices) there are examples of  $\Phi$  for which (2) does not hold with  $\hat{\Phi}$  replaced by any sentence whatsoever.

Suppose that  $\Phi(x)$  is trivial, for example suppose that  $\Phi(x)$  is the formula, " $x = x$ ". Then  $\hat{\Phi} \in T_\Phi$  if and only if there are no models

$$\mathcal{M} \models \text{ZFC}.$$

Therefore an immediate corollary of the lemma is that if there exists a model,

$$\mathcal{M} \models \text{ZFC},$$

then there exists a model,

$$\mathcal{N} \models \text{ZFC},$$

such that

$$\mathcal{N} \models \text{"There is no model of ZFC"}.$$

This of course is the version of Gödel's Second Incompleteness Theorem for the theory, ZFC.

The generic-multiverse position, which is suggested by the results of the previous section, declares that the Continuum Hypothesis is neither true nor false. Assuming that in each universe of the generic-multiverse there is a proper class of Woodin cardinals then the generic-multiverse position declares Projective Uniformization as true.

Is the generic-multiverse position a reasonable one? The refinements of Cohen's method of *forcing* in the decades since his initial discovery of the method

and the resulting plethora of problems shown to be unsolvable, have in a practical sense almost compelled one to adopt the generic-multiverse position. This has been reinforced by some rather unexpected consequences of large cardinal axioms which I shall discuss in the next section. Finally the argument that Cohen's method of forcing establishes that the Continuum Hypothesis has no answer, is implicitly assuming the generic-multiverse conception of truth, at least for sentences about sets of real numbers.

The purpose of this paper is *not* to argue against any possible multiverse position but to more carefully examine the generic-multiverse position within the context of modern Set Theory. I have formalized the generic-multiverse conception of truth simply as a vehicle to explore more fully the claim that Cohen's method of forcing does establish that the Continuum Hypothesis has no answer. In brief I shall argue that modulo the  $\Omega$  Conjecture, the generic-multiverse position outlined above is not reasonable and a more detailed discussion is given after a brief review of  $\Omega$ -logic, in the last two sections of this paper. The essence of the argument against the generic-multiverse position is that assuming the  $\Omega$  Conjecture is true (and that there is a proper class of Woodin cardinals) then this position is simply a brand of formalism that denies the transfinite by a reducing truth about the universe of sets to truth about a simple fragment such as the integers or, in this case, the collection of all subsets of the least Woodin cardinal. The  $\Omega$  Conjecture is invariant between  $V$  and  $V^{\mathbb{B}}$  and so the generic-multiverse position must either declare the  $\Omega$  Conjecture to be true or declare the  $\Omega$  Conjecture to be false.

It is a fairly common (informal) claim that the quest for truth about the universe of sets is analogous to the quest for truth about the physical universe. However I am claiming an important distinction. While physicists would rejoice in the discovery that the conception of the physical universe reduces to the conception of some simple fragment or model, in my view the set theorist must reject the analogous possibility for truth about the universe of sets. By the very nature of its conception, the set of all truths of the transfinite universe (the universe of sets) cannot be reduced to the set of truths of some explicit fragment of the universe of sets. Taking into account the iterative conception of sets, the set of all truths of an explicit fragment of the universe of sets cannot be reduced to the truths of an explicit *simpler* fragment. The latter is the basic position on which I shall base my arguments.

An assertion is  $\Pi_2$  if it is of the form,

$$\text{“ For every infinite ordinal } \alpha, V_\alpha \models \phi \text{ ”},$$

for some sentence,  $\phi$ . A  $\Pi_2$  assertion is a *multiverse truth* if the  $\Pi_2$  assertion holds in each universe of the multiverse.

Let  $\delta_0$  denote the least Woodin cardinal (so I now assume there is a proper class of Woodin cardinals so that the existence of  $\delta_0$  is invariant across the generic-multiverse).  $H(\delta_0^+)$  denotes the set of all sets  $X$  whose transitive

closure has cardinality at most  $\delta_0$ . The multiverse truths of  $H(\delta_0^+)$  are those sentences  $\phi$  which hold in the  $H(\delta_0^+)$  of each universe of the multiverse.

Note that for each sentence  $\phi$ , it is a  $\Pi_2$  assertion to say that

$$H(\delta_0^+) \models \phi$$

and it is a  $\Pi_2$  assertion to say that  $H(\delta_0^+) \not\models \phi$ . Thus in any one universe of the multiverse, the set of all sentences  $\phi$  such that

$$H(\delta_0^+) \models \phi,$$

this is the *theory* of  $H(\delta_0^+)$  as computed in that in that universe, is recursive in the set of  $\Pi_2$  sentences (assertions) which hold in that universe. Further by Tarski's Theorem on the undefinability of truth the latter set cannot be recursive in the former set.

Similarly as computed in any one universe of the multiverse, the theory of any *explicit* fragment of the universe of sets, such as  $V_{\omega+\omega}$  or even  $V_{\delta_0}$  is recursive in the set of  $\Pi_2$  sentences which hold in that universe and not vice-versa.

These comments suggest the following multiverse laws which I state in reference to an arbitrary multiverse position though in the context that the existence of a Woodin cardinal holds across the muliverse. For the case of the generic-multiverse generated by  $V$ , this latter requirement is equivalent to the requirement that there exist a proper class of Woodin cardinals in  $V$ .

### First Multiverse Law

*The set of  $\Pi_2$  assertions which are multiverse truths is not recursive in the set of multiverse truths of  $H(\delta_0^+)$ .*

The motivation for this multiverse law is that if the set of  $\Pi_2$  multiverse truths is recursive in the set of multiverse truths of  $H(\delta_0^+)$  then as far as evaluating  $\Pi_2$  assertions is concerned, the multiverse is equivalent to the reduced multiverse of just the fragments  $H(\delta_0^+)$  of the universes of the multiverse. This amounts to a rejection of the transfinite beyond  $H(\delta_0^+)$  and constitutes in effect the unacceptable brand of formalism alluded to earlier. This claim is reinforced should the multiverse position also violate a second multiverse law which I formulate below.

A set  $Y \subset V_\omega$  is definable in  $H(\delta_0^+)$  across the multiverse if the set  $Y$  is definable in the structure  $H(\delta_0^+)$  of each universe of the multiverse (possibly by formulas which depend on the parent universe).

The second multiverse law is a variation of the First Multiverse Law.

### Second Multiverse Law

*The set of  $\Pi_2$  assertions which are multiverse truths, is not definable in  $H(\delta_0^+)$  across the multiverse.*

Again, by Tarski's Theorem on the undefinability of truth, this multiverse law is obviously a reasonable one *if* one regards the only possibility for the multiverse to be the universe of sets so that set of multiverse truths of  $H(\delta_0^+)$  is simply the set of all sentences which are true in  $H(\delta_0^+)$  and the set of  $\Pi_2$  assertions which are multiverse truths is simply the set of  $\Pi_2$  assertions which are true in  $V$ . More generally the Second Multiverse Law would have to hold if one modified the law to simply require that the set of  $\Pi_2$  assertions which are multiverse truths, is not uniformly definable in  $H(\delta_0^+)$  across the multiverse (i.e. by a single formula).

Assuming both that  $\Omega$  Conjecture and the existence of a proper class of Woodin cardinals hold in each (or one) universe of the generic-multiverse generated by  $V$ , then *both* the First Multiverse Law and the Second Multiverse Law are violated by the generic-multiverse position. This is the basis for the argument I am giving against the generic-multiverse position in this paper. In fact the technical details of how the generic-multiverse position violates these multiverse laws provides an even more compelling argument against the generic-multiverse position since the analysis shows that in addition the generic-multiverse position is truly a form of formalism because of the connections to  $\Omega$ -logic.

There is a special case which I can present without any additional definitions and which is not contingent on any conjectures.

**THEOREM 9.** *Suppose that  $M$  is a countable transitive set*

$$M \models \text{ZFC} + \text{"There is a proper class of Woodin cardinals"}$$

*and that  $M \cap \text{Ord}$  is as small as possible. Then  $\mathbb{V}_M$  violates both multiverse laws.*

**§3.  $\Omega$ -logic.** A set  $X$  is *transitive* if for all  $a \in X$ ,  $a \subset X$ . It is a consequence of the axioms of Set Theory that every set  $X$  is a subset of a transitive set and among these transitive sets there is a least one under containment; this is the *transitive closure* of the set  $X$ .

A set  $X$  is of *hereditary cardinality at most  $\kappa$*  if the transitive closure of  $X$  has cardinality at most  $\kappa$ . As I indicated above, I denote by  $H(\delta_0^+)$  the set of all sets  $X$  whose transitive closure has cardinality at most  $\delta_0$  where  $\delta_0$  is the least Woodin cardinal.

Both the Continuum Hypothesis and Projective Uniformization are *first order* properties of  $H(\delta_0^+)$  in the sense that there are sentences  $\psi_{\text{CH}}$  and  $\psi_{\text{PU}}$  such that

$$H(\delta_0^+) \models \psi_{\text{CH}}$$

if and only if the Continuum Hypothesis holds and

$$H(\delta_0^+) \models \psi_{\text{PU}}$$

if and only if Projective Uniformization holds.

Since in the generic-multiverse position, an assertion of the form,

$$H(\delta_0^+) \models \phi,$$

is true if and only if the assertion holds in all universes of the generic-multiverse, the generic-multiverse position declares the Continuum Hypothesis to be neither true nor false and declares, granting large cardinals, that Projective Uniformization is true. I note that for essentially all current large cardinal axioms, the existence of a proper class of large cardinals holds in  $V$  if and only if it holds in  $V^{\mathbb{B}}$  for all complete Boolean algebras,  $\mathbb{B}$ . In other words, in the generic-multiverse position the existence of a proper class of, say, Woodin cardinals is either true or false since it either holds in every universe of the generic-multiverse or it holds in no universe of the generic-multiverse. [5].

I am going to analyze the generic-multiverse position from the perspective of  $\Omega$ -logic which I first briefly review.

**DEFINITION 10.** Suppose that  $T$  is a countable theory in the language of Set Theory, and  $\phi$  is a sentence. Then

$$T \models_{\Omega} \phi$$

if for all complete Boolean algebras,  $\mathbb{B}$ , for all ordinals,  $\alpha$ , if

$$V_{\alpha}^{\mathbb{B}} \models T$$

then  $V_{\alpha}^{\mathbb{B}} \models \phi$ .

If there is a proper class of Woodin cardinals then the relation  $T \models_{\Omega} \phi$ , is generically absolute. This fact which arguably was a completely unanticipated consequence of large cardinals, makes  $\Omega$ -logic interesting from a meta-mathematical point of view. For example the set

$$\mathcal{V}_{\Omega} = \{\phi \mid \emptyset \models_{\Omega} \phi\}$$

is generically absolute in the sense that for a given sentence,  $\phi$ , the question whether or not  $\phi$  is logically  $\Omega$ -valid, i.e. whether or not  $\phi \in \mathcal{V}_{\Omega}$ , is absolute between  $V$  and all of its generic extensions. In particular the method of (set) forcing *cannot* be used to show the formal independence of assertions of the form  $\emptyset \models_{\Omega} \phi$ .

**THEOREM 11.** *Suppose that  $T$  is a countable theory in the language of Set Theory, and  $\phi$  is a sentence. Suppose that there exists a proper class of Woodin cardinals. Then for all complete Boolean algebras,  $\mathbb{B}$ ,*

$$V^{\mathbb{B}} \models "T \models_{\Omega} \phi"$$

*if and only if  $T \models_{\Omega} \phi$ .*

There are a variety of technical theorems which show that one cannot hope to prove the generic invariance of  $\Omega$ -logic from any large cardinal hypothesis weaker than the existence of a proper class of Woodin cardinals—for example

if  $V = L$  then definition of  $\mathcal{V}_\Omega$  is not absolute between  $V$  and  $V^\mathbb{B}$ , for *any* nontrivial complete Boolean algebra,  $\mathbb{B}$ , of cardinality  $c$ .

It follows easily from the definition of  $\Omega$ -logic, that for any  $\Pi_2$ -sentence,  $\phi$ ,

$$\emptyset \models_\Omega \phi$$

if and only if for all complete Boolean algebras,  $\mathbb{B}$ ,

$$V^\mathbb{B} \models \phi.$$

Therefore by the theorem above, assuming there is a proper class of Woodin cardinals, for each sentence,  $\psi$ , the assertion

$$\text{For all complete Boolean algebras, } \mathbb{B}, V^\mathbb{B} \models "H(\delta_0^+) \models \psi"$$

is itself absolute between  $V$  and  $V^\mathbb{B}$  for all complete Boolean algebras  $\mathbb{B}$ . This remarkable consequence of the existence of a proper class of Woodin cardinals actually seems to be evidence for the generic-multiverse position. In particular this shows that the generic-multiverse position, at least for assessing  $\Pi_2$  assertions, and so for assessing all assertions of the form,

$$H(\delta_0^+) \models \phi,$$

is equivalent to the position that a  $\Pi_2$  assertion is true if and only if it holds in  $V^\mathbb{B}$  for all complete Boolean algebras  $\mathbb{B}$ . Notice that if  $\mathbb{R} \not\subset L$  and if  $V$  is a generic extension of  $L$  then this equivalence is *false*. In this situation the  $\Pi_2$  sentence which expresses  $\mathbb{R} \not\subset L$  holds in  $V^\mathbb{B}$  for all complete Boolean algebras,  $\mathbb{B}$ , but this sentence fails to hold across the generic-multiverse generated by  $V$  (since  $L$  belongs to this multiverse).

To summarize, suppose that there exists a proper class of Woodin cardinals in each universe of the generic-multiverse (or equivalently that there is a proper class of Woodin cardinals in at least one universe of the generic-multiverse). Then for each  $\Pi_2$  sentence  $\phi$ ; the following are equivalent:

- (1)  $\phi$  holds across the generic-multiverse;
- (2) " $\emptyset \models_\Omega \phi$ " holds across the generic-multiverse;
- (3) " $\emptyset \models_\Omega \phi$ " holds in at least one universe of the generic-multiverse.

Therefore to evaluate the generic-multiverse position one must understand the logical relation,  $T \models_\Omega \phi$ . In particular a natural question arises: is there a corresponding proof relation?

**§4. The  $\Omega$  conjecture.** I define the proof relation,  $T \vdash_\Omega \phi$ . This requires a preliminary notion that a set of reals be *universally Baire*, [2]. In fact I shall define  $T \vdash_\Omega \phi$ , assuming the existence of a proper class of Woodin cardinals and exploiting the fact that there are a number of (equivalent) definitions. Without the assumption that there is a proper class of Woodin cardinals, the definition is a bit more technical, [12]. Recall that if  $S$  is a compact Hausdorff

space then a set  $X \subseteq S$  has the *property of Baire* in the space  $S$  if there exists an open set  $O \subseteq S$  such that symmetric difference,

$$X \Delta O,$$

is meager in  $S$  (contained in a countable union of closed sets with empty interior).

DEFINITION 12. A set  $A \subset \mathbb{R}$  is *universally Baire* if for all compact Hausdorff spaces,  $S$ , and for all continuous functions,

$$F : S \rightarrow \mathbb{R},$$

the preimage of  $A$  by  $F$  has the property of Baire in the space  $S$ .

Suppose that  $A \subseteq \mathbb{R}$  is universally Baire. Suppose that  $M$  is a countable transitive model of ZFC. Then  $M$  is *strongly  $A$ -closed* if for all countable transitive sets  $N$  such that  $N$  is a generic extension of  $M$ ,

$$A \cap N \in N.$$

DEFINITION 13. Suppose there is a proper class of Woodin cardinals. Suppose that  $T$  is a countable theory in the language of Set Theory, and  $\phi$  is a sentence. Then  $T \vdash_{\Omega} \phi$  if there exists a set  $A \subset \mathbb{R}$  such that:

- (1)  $A$  is universally Baire,
- (2) for all countable transitive models,  $M$ , if  $M$  is strongly  $A$ -closed and  $T \in M$ , then

$$M \models "T \models_{\Omega} \phi".$$

Assuming there is a proper class of Woodin cardinals, the relation,  $T \vdash_{\Omega} \phi$ , is generically absolute. Moreover *Soundness* holds as well.

THEOREM 14. Assume there is a proper class of Woodin cardinals. Then for all  $(T, \phi)$  and for all complete Boolean algebras,  $\mathbb{B}$ ,

$$T \vdash_{\Omega} \phi \text{ if and only if } V^{\mathbb{B}} \models "T \vdash_{\Omega} \phi".$$

THEOREM 15 (Soundness). Assume there is a proper class of Woodin cardinals. If  $T \vdash_{\Omega} \phi$  then  $T \models_{\Omega} \phi$ .

I now come to the  $\Omega$  Conjecture which in essence is simply the conjecture that the Gödel Completeness Theorem holds for  $\Omega$ -logic; see [12] for a more detailed discussion.

DEFINITION 16 ( $\Omega$  Conjecture). Suppose that there exists a proper class of Woodin cardinals. Then for all sentences  $\phi$ ,  $\emptyset \models_{\Omega} \phi$  if and only if  $\emptyset \vdash_{\Omega} \phi$ .

**§5. The complexity of  $\Omega$ -logic.** Let (as defined on page 20)  $\mathcal{V}_\Omega$  be the set of sentences  $\phi$  such that

$$\emptyset \models_\Omega \phi,$$

and let  $\mathcal{V}_\Omega(H(\delta_0^+))$  be the set of sentences,  $\phi$ , such that

$$\text{ZFC} \models_\Omega "H(\delta_0^+) \models \phi".$$

Assuming there is a proper class of Woodin cardinals then the set of generic-multiverse truths which are  $\Pi_2$  assertions is of the same Turing complexity as  $\mathcal{V}_\Omega$  (i.e., each set is recursive in the other). Further (assuming there is a proper class of Woodin cardinals) the set,  $\mathcal{V}_\Omega(H(\delta_0^+))$ , is precisely the set of generic-multiverse truths of  $H(\delta_0^+)$ . Thus the requirement that the generic-multiverse position satisfies the First Multiverse Law, as discussed on page 24, reduces to the requirement that  $\mathcal{V}_\Omega$  not be recursive in the set  $\mathcal{V}_\Omega(H(\delta_0^+))$ .

The following theorem is a corollary of the basic analysis of  $\Omega$ -logic in the context that there is a proper class of Woodin cardinals, in fact one obtains the stronger conclusion that set  $\mathcal{V}_\Omega$  is recursive in the set  $\mathcal{V}_\Omega(H(\omega_2))$  (which is the set of sentences,  $\phi$ , such that  $\text{ZFC} \models_\Omega "H(\omega_2) \models \phi"$ ).

**THEOREM 17.** *Assume there is a proper class of Woodin cardinals and that the  $\Omega$  Conjecture holds. Then the set  $\mathcal{V}_\Omega$  is recursive in the set  $\mathcal{V}_\Omega(H(\delta_0^+))$ .*

Therefore, assuming the existence of a proper class of Woodin cardinals and that the  $\Omega$  Conjecture both hold across the generic-multiverse generated by  $V$ , the generic-multiverse position violates the First Multiverse Law. What about the Second Multiverse Law (on page 24)? This requires understanding the complexity of the set  $\mathcal{V}_\Omega$ . From the definition of  $\mathcal{V}_\Omega$  it is evident that this set is definable in  $V$  by a  $\Pi_2$  formula: if  $V = L$  then this set is recursively equivalent to the set of all  $\Pi_2$  sentences which are true in  $V$ . However in the context of large cardinal axioms the complexity of  $\mathcal{V}_\Omega$  is more subtle.

**THEOREM 18.** *Assume there is a proper class of Woodin cardinals and that the  $\Omega$  Conjecture holds. Then the set  $\mathcal{V}_\Omega$  is definable in  $H(\delta_0^+)$ .*

Therefore if the  $\Omega$  Conjecture holds and there is a proper class of Woodin cardinals then the generic-multiverse position that the only  $\Pi_2$  assertions which are true are those which are true in each universe of the generic-multiverse also violates the Second Multiverse Law—for this set of assertions is itself definable in  $H(\delta_0^+)$  across the generic-multiverse.

In summary, assuming the existence of a proper class of Woodin cardinals and that the  $\Omega$  Conjecture both hold across the generic-multiverse generated by  $V$ , then both the First Multiverse Law and the Second Multiverse Law are violated by the generic-multiverse view of truth.

In particular, assuming the existence of a proper class of Woodin cardinals and that the  $\Omega$  Conjecture both hold across the generic-multiverse generated by  $V$ , then as far evaluating truth across the generic-multiverse for  $\Pi_2$  assertions, the generic-multiverse is equivalent to the reduced multiverse given by the

structures  $H(\delta_0^+)$  of the universes in the generic-multiverse. Why not just adopt formalism where the multiverse is the collection of all possible universes constrained only by the formal axioms, ZFC, so that truth is reduced to truth within  $V_\omega$  (i.e. to truth within the integers)?

As I have indicated, the actual argument against the generic-multiverse position is a much more compelling one. This is because the reduction of truth for  $\Pi_2$  assertions to truth about  $H(\delta_0^+)$  is much stronger than is abstractly indicated by the mere violation of the two multiverse laws. It seems incoherent to me to have a conception of the transfinite which reduces to simply a conception of  $H(\delta_0^+)$  which in essence is just the truncation of the universe of sets to the level of the least Woodin cardinal.

**§6. The weak multiverse laws and  $H(c^+)$ .** The standard structure for *Third Order Number Theory* is the structure,  $\langle \mathcal{P}(\mathbb{R}), \mathbb{R}, \cdot, +, \in \rangle$ . This structure is logically equivalent to  $H(c^+)$  which is the set of all sets  $X$  whose transitive closure has cardinality at most  $c = 2^{\aleph_0}$ .

The multiverse truths of  $H(c^+)$  are those sentences  $\phi$  which hold in the  $H(c^+)$  of each universe of the multiverse. The following are the natural formally weaker versions of the two multiverse laws which are obtained by simply replacing  $H(\delta_0^+)$  by  $H(c^+)$ . For the Second Multiverse Law, this gives the weakest version which is not provable for the generic-multiverse assuming the existence of a proper class of Woodin cardinals; i.e., replacing  $H(c^+)$  by  $H(c)$  yield a second multiverse law which is provable for the generic-multiverse assuming the existence of a proper class of Woodin cardinals.

#### Weak First Multiverse Law

*The set of  $\Pi_2$  assertions which are multiverse truths is not recursive in the set of multiverse truths of  $H(c^+)$ .*

#### Weak Second Multiverse Law

*The set of  $\Pi_2$  assertions which are multiverse truths, is not definable in  $H(c^+)$  across the multiverse.*

The following theorem shows that assuming there exist a proper class of Woodin cardinals and that the  $\Omega$  Conjecture holds, then the generic-multiverse position violates the Weak First Multiverse Law.

**THEOREM 19.** *Assume there is a proper class of Woodin cardinals and that the  $\Omega$  Conjecture holds. Then the set  $\mathcal{V}_\Omega$  is recursive in the set  $\mathcal{V}_\Omega(H(c^+))$ .*

Theorem 9 also holds for the weak multiverse laws.

**THEOREM 20.** *Suppose that  $M$  is a countable transitive set*

$M \models \text{ZFC} + \text{“There is a proper class of Woodin cardinals”}$

and that  $M \cap \text{Ord}$  is as small as possible. Then  $\mathbb{V}_M$  violates both weak multiverse laws.

The issue of whether the generic-multiverse position violates the Weak Second Multiverse Law is much more subtle and the issue is whether assuming the  $\Omega$  Conjecture one can show that  $\mathcal{V}_\Omega$  is definable in  $H(c^+)$  across the generic-multiverse. I originally thought I could prove this but the proof was based on an implicit restriction regarding the universally Baire sets, see [14] for more details.

The definability of  $\mathcal{V}_\Omega$  in  $H(c^+)$  is a consequence of the  $\Omega$  Conjecture augmented by the following conjecture. The statement involves both  $\text{AD}^+$  which is a technical variant of  $\text{AD}$ , the *Axiom of Determinacy*, and the notion that a set  $A \subseteq \mathbb{R}$  be  $\omega_1$ -universally Baire. The definition of  $\text{AD}^+$  and a brief survey of some of the basic aspects of the theory of  $\text{AD}^+$  are given in [14].

Suppose that  $\delta$  is an infinite cardinal. A set  $A \subset \mathbb{R}$  is  $\delta$ -universally Baire if for all compact Hausdorff spaces  $\Omega$  and for all continuous functions

$$\pi : \Omega \rightarrow \mathbb{R}$$

if the topology of  $\Omega$  has a basis of cardinality  $\kappa$  for some  $\kappa \leq \delta$ , then the preimage of  $A$  by  $\pi$  has the property of Baire in  $\Omega$ . Clearly  $A$  is universally Baire if and only if  $A$  is  $\delta$ -universally Baire for all  $\delta$ .

**DEFINITION 21** ( $\text{AD}^+$  Conjecture). Suppose that  $L(A, \mathbb{R})$  and  $L(B, \mathbb{R})$  each satisfy  $\text{AD}^+$ . Suppose that every set

$$X \in (L(A, \mathbb{R}) \cup L(B, \mathbb{R})) \cap \mathcal{P}(\mathbb{R})$$

is  $\omega_1$ -universally Baire. Then either

$$(\mathbb{A}_1^2)^{L(A, \mathbb{R})} \subseteq (\mathbb{A}_1^2)^{L(B, \mathbb{R})}$$

or

$$(\mathbb{A}_1^2)^{L(B, \mathbb{R})} \subseteq (\mathbb{A}_1^2)^{L(A, \mathbb{R})}$$

There is a stronger version of this conjecture.

**DEFINITION 22** (strong  $\text{AD}^+$  conjecture). Suppose that  $L(A, \mathbb{R})$  and  $L(B, \mathbb{R})$  each satisfy  $\text{AD}^+$ . Suppose that every set

$$X \in (L(A, \mathbb{R}) \cup L(B, \mathbb{R})) \cap \mathcal{P}(\mathbb{R})$$

is  $\omega_1$ -universally Baire. Then either  $A \in L(B, \mathbb{R})$  or  $B \in L(A, \mathbb{R})$ .

Assuming the  $\text{AD}^+$  Conjecture one obtains a significant improvement on the calculation of the complexity of  $\mathcal{V}_\Omega$  and here, as opposed to the previous theorem, the distinction between  $H(c^+)$  and  $H(\omega_2)$  is critical.

**THEOREM 23** ( $\text{AD}^+$  Conjecture). Assume there is a proper class of Woodin cardinals. Then the set

$$\mathcal{V}_\Omega = \{\phi \mid \emptyset \models_\Omega \phi\}$$

is definable in  $H(c^+)$ .

Thus assuming the  $\text{AD}^+$  Conjecture holds across the generic-multiverse then the generic-multiverse position violates the Weak Second Multiverse Law assuming of course that both the existence of a proper class of Woodin cardinals and the  $\Omega$  Conjecture hold in  $V$ . Quite a number of combinatorial propositions are known to imply the  $\text{AD}^+$  Conjecture [13], and the results of [14] offer evidence that the  $\text{AD}^+$  Conjecture is true.

For my basic argument against the generic-multiverse position (assuming the  $\Omega$  Conjecture), arguably the distinction between  $H(c^+)$  and  $H(\delta_0^+)$  is not relevant. It is however crucial for arguments such as those given in [11] that the Continuum Hypothesis is false. Those arguments are now contingent on *both* the  $\Omega$  Conjecture and the  $\text{AD}^+$  Conjecture. Nevertheless the results of [14] strongly suggest that these conjectures are both true. However the results of [14] also suggest that Continuum Hypothesis is true. This is not a contradiction. The basic argument against the Continuum Hypothesis in [11] is based on optimizing the theory of  $H(\omega_2)$  and this approach cannot extend to even  $H(c^+)$ . While this approach is perhaps compelling from the perspective of just  $H(\omega_2)$ , it now seems likely that it will not be so compelling from the perspective of  $V$ .

**§7. Conclusions.** If the  $\Omega$  Conjecture is true then one cannot reasonably claim that the only true  $\Pi_2$  assertions are those which are true across the generic-multiverse. Of course I am assuming large cardinals exist, in particular I am assuming that in some (and hence all) universes of the generic-multiverse, there is a proper class of Woodin cardinals.

For the skeptic who acknowledges this and yet claims that the problem of the Continuum Hypothesis is meaningless the challenge is the following:

*Exhibit a  $\Pi_2$  assertion which is true and which is not true across the generic-multiverse.*

The notion,  $T \vdash_{\Omega} \phi$ , has many features of the classical notion,  $T \vdash \phi$ . These features include a reasonable definition for the length of proof and so one can construct Gödel and Rosser sentences within  $\Omega$ -logic. Moreover if the  $\Omega$  Conjecture holds then  $\mathcal{V}_{\Omega}$  is definable in  $H(\delta_0^+)$ . Thus if the  $\Omega$  Conjecture holds then meeting the challenge posed above would seem to be entirely straightforward, being analogous to the challenge of exhibiting a sentence  $\psi$  for which the assertion

$$V_{\omega} \models \psi$$

is both true and not provable (in first order logic). In fact it would seem one could do much better and exhibit an assertion of the form

$$H(\delta_0^+) \models \phi$$

which is true but not true across the generic-multiverse.

However there is a feature of  $\Omega$  logic which is not shared by classical logic. While it is true that if the  $\Omega$  Conjecture holds then  $\mathcal{V}_\Omega$  is definable in  $H(\delta_0^+)$ , the definition is *not uniform*. More precisely the actual definition of  $\mathcal{V}_\Omega$  within  $H(\delta_0^+)$  depends on the universe  $V$ . A strong form of this claim is given by the following theorem.

**THEOREM 24.** *Assume there is a proper class of Woodin cardinals and that the  $\Omega$  Conjecture holds. Then for each formula  $\Psi(x)$  there exists a complete Boolean algebra  $\mathbb{B}$  such that*

$$V^{\mathbb{B}} \models \text{“}\mathcal{V}_\Omega \neq \{\phi \mid H(\delta_0^+) \models \Psi[\phi]\}\text{”}.$$

Thus, assuming there is a proper class of Woodin cardinals and that the  $\Omega$  Conjecture holds, one *cannot* use either Gödel or Rosser style sentences to produce an assertion of the form

$$H(\delta_0^+) \models \psi$$

which is true and not  $\Omega$ -valid; where a  $\Pi_2$  assertion is  $\Omega$ -valid if it holds in  $V^{\mathbb{B}}$  for all complete Boolean algebras,  $\mathbb{B}$ . Any such construction would have to be based on a specific choice of the definition of  $\mathcal{V}_\Omega$  within  $H(\delta_0^+)$ ; and there is no possible choice which one can make.

A similar argument applies to attempting to meet the challenge stated above—producing a  $\Pi_2$  assertion which is true but not  $\Omega$ -valid—because of the following stronger version of the previous theorem.

**THEOREM 25.** *Assume there is a proper class of Woodin cardinals and that the  $\Omega$  Conjecture holds. Then for each  $\Sigma_2$  formula  $\Psi(x)$  there exists a complete Boolean algebra  $\mathbb{B}$  such that*

$$V^{\mathbb{B}} \models \text{“}\mathcal{V}_\Omega \neq \{\phi \mid \Psi[\phi] \text{ holds}\}\text{”}.$$

Of course one could just conclude from all of this that the  $\Omega$  Conjecture is *false*; see [12] for a more detailed discussion of this; declaring the  $\Omega$  Conjecture to be meaningless is *not* an option since the  $\Omega$  Conjecture is either true in all the universes of the generic-multiverse or false in all the universes of the generic-multiverse.

Note that even if the  $\Omega$  Conjecture is false this does not necessarily resolve the objections to generic-multiverse position. For this end one would need something much stronger than the simple failure of the  $\Omega$  Conjecture; one would need at the very least either that  $\mathcal{V}_\Omega$  is not recursive in  $\mathcal{V}_\Omega(H(\delta_0^+))$  or that  $\mathcal{V}_\Omega$  is not definable in  $H(\delta_0^+)$  across the generic-multiverse. If the  $\Omega$  Conjecture is false then the problem of trying to understand the complexity of  $\mathcal{V}_\Omega$  looks extremely difficult. For example, is it consistent for there to be a proper class of Woodin cardinals and for  $\mathcal{V}_\Omega$  to be recursively equivalent to the set of all  $\Pi_2$ -sentences which hold in  $V$  (as happens if  $V = L$ ) and so be as complicated as its natural definition suggests?

The skeptic might try a different approach even granting the  $\Omega$  Conjecture and that there exists proper class of Woodin cardinals, by proposing that the generic-multiverse is *too small*. In particular why should a  $\Pi_2$  assertion which is  $\Omega$ -valid in *one* universe of the multiverse be true in *every* universe of the multiverse?

The counter to this approach is simply that, since the  $\Omega$  Conjecture and the existence of a proper class of Woodin cardinals both hold across the multiverse, if “ $\emptyset \vdash_{\Omega} \phi$ ” holds in one universe of the multiverse then “ $\emptyset \vdash_{\Omega} \phi$ ” holds in that universe. But then “ $\emptyset \vdash_{\Omega} \phi$ ” must hold across the multiverse and so by the Soundness Theorem for  $\Omega$ -logic, the sentence  $\phi$  must hold in every universe of the multiverse. The argument that “ $\emptyset \vdash_{\Omega} \phi$ ” must hold across the multiverse is made by appealing to the intricate connections between proofs in  $\Omega$ -logic and notions of large cardinals; so one is really arguing that however the multiverse is defined, each universe of the multiverse is *as transcendent* as every other universe of the multiverse. Of course the  $\Omega$  Conjecture is essential for this argument though interestingly the use of the  $\Omega$  Conjecture is quite different here.

The  $\Omega$  Conjecture is consistent (with a proper class of Woodin cardinals), more precisely if the theory

$$\text{ZFC} + \text{“There is a proper class of Woodin cardinals”}$$

is consistent then so is this theory together with the assertion that the  $\Omega$  Conjecture is true. It is at present not known if the  $\Omega$  Conjecture is consistently false.

The *Inner Model Program* is the attempt to generalize the definition of Gödel’s constructible universe,  $L$ , to define (canonical) transitive inner models of the universe of sets for *large cardinal axioms*. Fairly general requirements on the structure of the inner model imply that the  $\Omega$  Conjecture must hold in the inner model, further very recent results indicate that if this program can succeed at the level of supercompact cardinals then no large cardinal hypothesis whatsoever can refute the  $\Omega$  Conjecture. Such an analysis would in turn strongly suggest that the  $\Omega$  Conjecture is true; [14]. In summary there is (at present) a plausible framework for actually proving the  $\Omega$  Conjecture—of course evidence is not a proof and this framework could collapse under an onslaught of theorems which reveal the true nature of sets.

But suppose that the  $\Omega$  Conjecture is in fact provable (in classical logic, from the axioms for Set Theory). After all, as indicated above, all the evidence to date points to this possibility. What would this say about truth within Set Theory? It certainly would say that there is no evidence for the claim that the Continuum Hypothesis has no answer.

If the  $\Omega$  Conjecture holds then there *must* be a  $\Pi_2$  assertion which is true and which is not  $\Omega$ -valid—in fact for essentially the same reasons there must

be an assertion of the form,

$$H(\delta_0^+) \models \phi$$

for some sentence  $\phi$ , which is true but not  $\Omega$ -valid.

The latter class of  $\Pi_2$  assertions,  $\Phi$ , have the feature that they can be equivalently formulated as  $\Sigma_2$  assertions—in the sense that there is a  $\Sigma_2$  sentence,  $\Phi'$ , with the property that in each universe of the generic-multiverse generated by  $V$ ,  $\Phi$  holds if and only if  $\Phi'$  holds. Any such  $\Pi_2$  assertion,  $\Phi$ , is qualitatively just like both the *Continuum Hypothesis* and its negation—assuming there is a proper class of Woodin cardinals there are complete Boolean algebras,  $\mathbb{B}$ , in which the assertion holds as interpreted in  $V^{\mathbb{B}}$  and there are complete Boolean algebras,  $\mathbb{B}$ , in which the assertion fails as interpreted in  $V^{\mathbb{B}}$ . So if there is such a sentence  $\Phi$  which is true then why could this not also be the case for Continuum Hypothesis (or its negation)?

I just do not see how one can maintain a position that there is any meaning to a conception of the transfinite universe beyond formalism, and yet be unwilling to acknowledge that there is some statement about  $H(\delta_0^+)$  which is both true and not  $\Omega$ -valid (unless the  $\Omega$  Conjecture is false).

But again the skeptic can reasonably object: OK, even if the Continuum Hypothesis has an answer, is there any evidence whatsoever that we can or will ever determine what that answer is? The sympathetic skeptic might soften the position implicit in this question and simply claim that we are as far from finding and understanding the answer to the problem of the Continuum Hypothesis as the mathematicians studying the projective sets in the early 20th century were from finding and understanding the answer to the problem of Projective Uniformization.

Perhaps the generic-multiverse position can be (non-trivially) resurrected by adding single sentence to the axioms, ZFC, and still assuming both the  $\Omega$  Conjecture and the existence of a proper class of Woodin cardinals both hold across the multiverse. Adding a sentence,  $\Psi$ , to the axioms, ZFC, one would restrict the multiverse to a collection of universes where  $\Psi$  holds. Applying this restriction to the generic-multiverse gives the corresponding restricted generic-multiverse leading to a *revised generic-multiverse position*. This can be formalized within in any universe of the restricted generic-multiverse; Lemma 33, just as in the case for the generic-multiverse position. Since by adding a  $\Pi_2$  sentence to the axioms one can preserve the truth of any given  $\Sigma_2$  sentence, preserve essentially all large cardinals, and force the corresponding restricted generic-multiverse to contain only one universe, I shall restrict consideration to the case where the additional axiom is a  $\Sigma_2$  sentence.

Suppose  $\Psi$  is a sentence and let  $\mathcal{V}_{\Omega}^{\Psi}$  be the set of sentences,  $\phi$ , such that

$$\{\Psi\} \models_{\Omega} \phi,$$

and let  $\mathcal{V}_\Omega^\Psi(H(\delta_0^+))$  be the set of sentences,  $\phi$ , such that

$$\text{ZFC} \cup \{\Psi\} \models_\Omega "H(\delta_0^+) \models \phi".$$

Now if  $\Psi$  is a  $\Sigma_2$  sentence then  $\mathcal{V}_\Omega^\Psi$  is of the same Turing complexity as the set of  $\Pi_2$  assertions which hold across the restricted generic-multiverse (where now  $\Psi$  is required to hold in each universe of the multiverse) and  $\mathcal{V}_\Omega^\Psi(H(\delta_0^+))$  is the corresponding set of multiverse truths of  $H(\delta_0^+)$ .

Clearly  $\mathcal{V}_\Omega^\Psi$  is recursive in  $\mathcal{V}_\Omega$  and so by Theorem 18, the revised generic-multiverse position will still violate the Second Multiverse Law. What about the First Multiverse Law?

**THEOREM 26.** *Assume there is a proper class of Woodin cardinals and that the  $\Omega$  Conjecture holds. Then for each sentence  $\Psi$ , the set  $\mathcal{V}_\Omega^\Psi$  is recursive in the set  $\mathcal{V}_\Omega^\Psi(H(\delta_0^+))$ .*

Thus assuming both the  $\Omega$  Conjecture and the existence of a proper class of Woodin cardinals both hold across the multiverse then for all enlargements of ZFC by adding a single  $\Sigma_2$  sentence to the axioms, ZFC, the revised generic-multiverse position still violates both the First Multiverse Law and the Second Multiverse Law. Assuming in addition that the  $\text{AD}^+$  Conjecture holds across the generic-multiverse this violation extends to both the Weak First Multiverse Law and the Weak Second Multiverse Law. Though here as above, the distinction between  $H(\omega_2)$  and  $H(c^+)$  is critical.

I am an optimist, perhaps even a transfiniteist. There is in my view no reason at all, beyond a lack of faith, for believing that there is no extension of the axioms ZFC, by one axiom, a posteriori true, which settles all instances of the Generalized Continuum Hypothesis and more generally which yields a theory of the universe of sets which is as “complete” as the theory of Gödel’s constructible universe,  $L$ , which is given by the axioms ZFC. (Or as complete as the theory of the integers that is given by the axioms for that structure.) The new axiom of any such extension cannot be  $\Omega$ -valid since in particular it must settle the Continuum Hypothesis.

Rephrasing my position slightly, there is in my view no credible evidence at present refuting the existence of a single additional axiom to ZFC which is consistent with large cardinal axioms and which in a practical sense provides a complete description of  $H(\delta_0^+)$  or even of  $V_\kappa$  where  $\kappa$  is any cardinal which is definable within the universe of sets as the least cardinal with a  $\Sigma_2$ -property; where the gold standard for practical completeness is the theory of  $L$  as given by the ZFC axioms.

Until recently I have always viewed such a possibility as very implausible at best. The change in my view is motivated by the results of [14] which provide some evidence for the existence of such an axiom. This is not to say that in the final analysis I will not revert.

Why not the axiom “ $V = L$ ”? The difficulty is that this is a limiting axiom for it refutes large cardinal axioms. The results of [14] suggest the possibility that if there is a *supercompact cardinal* then there is a generalization of  $L$  which is both *close* to  $V$  and which inherits large cardinals from  $V$  exactly as  $L$  inherits large cardinals from  $V$  if  $0^\#$  does not exist. Should this turn out to be true, it would be remarkable. The axiom that  $V$  is such an inner model would have all the advantages of the axiom “ $V = L$ ” without limiting  $V$  as far as large cardinals are concerned. In particular the often cited arguments against the axiom “ $V = L$ ” *would not apply to this new axiom*.

A far stronger view than that outlined above and which I also currently hold because of the suggestive results of [14], is that there *must* be such an axiom and in understanding it we will understand why it is essentially unique and therefore true. Further this new axiom will in a transparent fashion both settle the classical questions of combinatorial set theory where to date independence has been the rule and explain the large cardinal hierarchy. There is already a specific candidate for this axiom [14] though it is not only too early to argue that this axiom is true, it is too early to be sure that this axiom is even consistent with all large cardinal axioms. The issue at present is that there are actually *two* families of enlargements of  $L$  and it is not yet clear whether *both* can be *transcendent* relative to large cardinal axioms, [14].

Even if this does happen (we find and understand this missing axiom), the specter of independence remains—it is just that now the vulnerability of Set Theory to the occurrence of independence becomes the same as that of Number Theory. In other words, we would have come to a conception of the transfinite universe which is as clear and unambiguous as our conception of the fragment  $V_\omega$ , the universe of the finite integers. To me this a noteworthy goal to aspire to.

**§8. Appendix.** The purpose of this appendix is to show how the generic-multiverse position can be formalized within  $V$ . In particular I shall prove that for each sentence  $\phi$  there is a sentence  $\phi^*$ , recursively depending on  $\phi$ , such that for each countable transitive set  $M$  such that  $M \models \text{ZFC}$ , the following are equivalent:

- (1)  $M \models \phi^*$ ;
- (2) For each  $N \in \mathbb{V}_M$ ,  $N \models \phi$ ;

where as defined on page 15,  $\mathbb{V}_M$  is the generic-multiverse generated (in  $V$ ) by  $M$ . In fact this is a straightforward corollary of Lemma 27 and Lemma 28 below noting that to verify that  $N \models \phi$  for each  $N \in \mathbb{V}_M$  one need only verify  $N \models \phi$  for each transitive set  $N$  such that  $N$  can be generated from  $M$  in only 3 steps (actually 2 steps as noted by Hamkins) of taking generic enlargements or generic refinements (as opposed to finitely many steps). However the proof I shall give easily adapts to prove the corresponding result for *any* restricted

generic-multiverse position obtained by limiting the generic extensions to those given by a definable class of partial orders, Lemma 33. In this general case, the reduction to models generated in only 3 steps (or any fixed finite number of steps) from the initial model  $M$  is not always possible.

Let  $ZC^{(VN)}$  denote the axioms  $ZC$  together with the axiom which asserts that for all ordinals  $\alpha$ ,  $V_\alpha$  exists.

I fix some notation generalizing the definition of  $\mathbb{V}_M$  to the case that  $M$  is a countable transitive set such that

$$M \models ZC^{(VN)} + \Sigma_1\text{-Replacement}.$$

Note that

$$V_{\omega+\omega} \models ZC^{(VN)},$$

but for all (ordinals)  $\lambda > \omega$ ,

$$V_\lambda \models ZC^{(VN)} + \Sigma_1\text{-Replacement},$$

if and only if  $|V_\lambda| = \lambda$ .

Let  $\mathbb{V}_M$  be the generic-multiverse generated by  $M$ . This is the smallest collection of transitive sets such that  $M \in \mathbb{V}_M$  and such that for all pairs,  $(M_1, M_2)$ , of countable transitive sets if

- (1)  $M_1 \models ZC^{(VN)} + \Sigma_1\text{-Replacement}$ ,
- (2)  $M_2$  is a generic extension of  $M_1$ ,
- (3) either  $M_1 \in \mathbb{V}_M$  or  $M_2 \in \mathbb{V}_M$ ,

then both  $M_1$  and  $M_2$  are in  $\mathbb{V}_M$ .

The only change here from the definition of  $\mathbb{V}_M$  on page 15, is that here I am not requiring that  $M \models ZFC$ ; but just that  $M$  be a model of the weaker set of axioms,

$$ZC^{(VN)} + \Sigma_1\text{-Replacement}.$$

The following lemma has an interesting corollary. Suppose  $M$  is a transitive set,  $M \models ZFC$ , and that  $M[G]$  is a generic extension of  $M$ . Then  $M$  is definable in  $M[G]$  from parameters. This in turn implies that the property of being a generic extension is first-order. The lemma is motivated by [4] and both a version of this lemma and the application indicated above are due independently to Laver, [6]; also see Reitz [10] for further developments.

LEMMA 27. *Suppose that  $M, M'$  are transitive sets,*

$$\mathbb{P} \in M \cap M'$$

*and  $G \subset \mathbb{P}$ . Suppose:*

- (i)  $M \models ZC^{(VN)} + \Sigma_1\text{-Replacement}$ ;
- (ii)  $M' \models ZC^{(VN)} + \Sigma_1\text{-Replacement}$ ;
- (iii)  $\mathcal{P}(\mathbb{P}) \cap M = \mathcal{P}(\mathbb{P}) \cap M'$ ;

- (iv)  $G$  is  $M$ -generic for  $\mathbb{P}$ ;
- (v)  $M[G] = M'[G]$ .

Then  $M = M'$ .

PROOF. Suppose toward a contradiction that  $M \neq M'$ . Then since

$$M \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement},$$

it follows that  $M \cap \mathcal{P}(\text{Ord}) \neq M' \cap \mathcal{P}(\text{Ord})$ .

Let  $\gamma \in \text{Ord} \cap M$  be least such that

$$\mathcal{P}(\gamma) \cap M \neq \mathcal{P}(\gamma) \cap M'.$$

Clearly  $\gamma$  is a cardinal in both  $M$  and  $M'$ . Let  $\kappa = |\mathbb{P}|^M$ . Thus  $(\kappa^+)^M$  is a cardinal in  $M[G]$  and since,

$$\mathcal{P}(\mathbb{P}) \cap M = \mathcal{P}(\mathbb{P}) \cap M',$$

it follows that  $\gamma > \kappa$ . By interchanging  $M$  and  $M'$  if necessary we can suppose that there exists a set

$$A \in \mathcal{P}(\gamma) \cap M' \setminus M.$$

Thus by choice of  $\gamma$ , for all  $\alpha < \gamma$ ,  $A \cap \alpha \in M$ . Further since  $A \in M[G]$ , it follows that

$$(\text{cof}(\gamma))^M \leq \kappa.$$

Therefore  $\gamma$  is not a regular cardinal in  $M$  and so  $\gamma > (\kappa^+)^M$ . Let

$$\delta = (\kappa^+)^M = (\kappa^+)^{M'} = (\kappa^+)^{M[G]}.$$

The key point is that in  $M[G]$  the following hold.

- (1.1)  $\mathcal{P}(\gamma) \cap M'$  is closed in under strictly increasing unions of length  $\delta$ .
- (1.2) For every set  $a \subset \gamma$  with  $|a|^{M[G]} \leq \delta$ , there exists  $b \subset \gamma$  such that  $a \subseteq b$ ,  $|b|^{M[G]} \leq \delta$ , and such that  $b \in \mathcal{P}(\gamma) \cap M'$ .
- (1.3) For each  $\xi < \gamma$ ,  $\mathcal{P}(\xi) \cap M' = \mathcal{P}(\xi) \cap M$ .
- (1.4)  $A \in \mathcal{P}(\gamma) \cap M' \setminus M$ .

Let  $\tau \in M$  be a term for  $\mathcal{P}(\gamma) \cap M'$ . Let  $p \in G$  be a condition which forces that (1.1)–(1.3) hold for  $I_G(\tau)$ , where  $I_G(\tau)$  is the interpretation of  $\tau$  in  $M[G]$ . Let  $\sigma \in M$  be a term for  $A$ . By shrinking  $p$  if necessary we can suppose that  $p$  forces both  $I_G(\sigma) \in I_G(\tau)$  and that  $I_G(\sigma) \notin M$ .

We now work in  $M$ . Let

$$X = \{Z \subset \gamma \mid \kappa \subset Z, |Z| \leq \delta\}$$

and let  $S$  be the set of all  $Z \in X$ , such that  $p$  forces  $Z \in I_G(\tau)$ . Since  $p$  forces that (1.1)–(1.2) hold for  $I_G(\tau)$ , it follows that  $S$  is stationary as a subset of  $\{Z \subset \gamma \mid |Z| \leq \delta\}$ . To see this let

$$H : \gamma^{<\omega} \rightarrow \gamma.$$

We must find  $Z \in S$  such that  $H[Z^{<\omega}] \subset Z$ . Let

$$\langle (Z_\alpha, \tau_\alpha) : \alpha < \delta \rangle$$

be a sequence such that for all  $\alpha < \beta < \delta$ ,

- (2.1)  $p \Vdash \tau_\alpha \in \tau$ ,
- (2.2)  $p \Vdash \tau_\alpha \subseteq Z_\beta$ ,
- (2.3)  $p \Vdash Z_\alpha \subseteq \tau_\beta$ ,
- (2.4)  $|Z_\alpha| \leq \delta$  and  $H[Z_\alpha^{<\omega}] \subset Z_\alpha$ .

Such a sequence is easily constructed by induction. Notice that for all  $\alpha < \beta$ ,  $p \Vdash \tau_\alpha \subseteq \tau_\beta$ . Thus

$$p \Vdash \bigcup \{\tau_\alpha \mid \alpha < \delta\} \in \tau.$$

But letting  $Z = \bigcup \{Z_\alpha \mid \alpha < \delta\}$ ,

$$p \Vdash \bigcup \{\tau_\alpha \mid \alpha < \delta\} = Z,$$

and so  $Z \in S$ . Clearly  $H[Z^{<\omega}] \subset Z$  and this shows that  $S$  is stationary.

For each  $Z \in S$ , there exist  $p_Z \leq p$  and  $A_Z \subseteq Z$  such that  $p_Z$  forces that

$$I_G(\sigma) \cap Z = A_Z.$$

This is because  $p$  forces that (1.3)–(1.4) holds for  $I_G(\tau)$ . But  $|\mathbb{P}|^M = \kappa$  and  $\kappa \subset Z$  for each  $Z \in S$ . Therefore there exists  $q \leq p$  such that

$$S_q = \{Z \in S \mid p_Z = q\}$$

is stationary as a subset of  $\{Z \subset \gamma \mid |Z| \leq \delta\}$ . For each  $Z_1, Z_2 \in S_q$ , it follows that

$$A_{Z_1} \cap Z_1 \cap Z_2 = A_{Z_2} \cap Z_1 \cap Z_2.$$

Thus there exists a set  $A_q \subset \gamma$  such that for all  $Z \in S_q$ ,  $A_q \cap Z = A_Z$ . But then  $q$  forces  $I_G(\sigma) = A_q$  which contradicts the choice of  $\sigma$  and  $p$ .

Therefore  $M \cap \mathcal{P}(\text{Ord}) = M' \cap \mathcal{P}(\text{Ord})$  and so  $M = M'$ .  $\dashv$

The next lemma is a corollary of Lemma 27.

LEMMA 28. *Suppose that  $N$  is transitive, the set*

$$\left\{ \xi \in N \cap \text{Ord} \mid N \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement} \right\},$$

*is cofinal in  $N \cap \text{Ord}$ ,  $\alpha \in N \cap \text{Ord}$ , and that  $N_\alpha \prec_{\Sigma_2} N$ .*

*Suppose that  $M \in N$  is a transitive set and:*

- (i)  $M \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement}$ ;
- (ii)  $\mathbb{P} \in M$ ,  $G \subseteq \mathbb{P}$ , and  $G$  is  $M$ -generic for  $\mathbb{P}$ ;
- (iii)  $G \in N$ ;
- (iv)  $M[G] = N \cap V_\alpha$ .

*Then there exists a transitive set  $M^* \subset N$  such that*

- (1)  $M^* \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement}$ ,
- (2)  $M^* \cap V_\alpha = M$ ,
- (3)  $N = M^*[G]$ .

PROOF. Let  $I$  be the set of  $\xi \in N \cap \text{Ord}$  such that

$$N \cap V_\xi \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement}$$

and such that  $\mathbb{P} \in N \cap V_\xi$ .

Thus  $\alpha \in I$  and both the hypothesis on  $N$ ,

(1.1)  $I \cap \alpha$  is cofinal in  $\alpha$ ,

(1.2)  $I$  is cofinal in  $N \cap \text{Ord}$ .

Let  $I'$  be the set of  $\xi \in I$  such that there exists a transitive set  $M' \in N$  such that

(2.1)  $M' \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement}$ ,

(2.2)  $\mathbb{P} \in M'$ ,

(2.3)  $\mathcal{P}(\mathbb{P}) \cap M' = \mathcal{P}(\mathbb{P}) \cap M$ ,

(2.4)  $N \cap V_\xi = M'[G]$ .

Clearly  $\alpha \in I'$  and  $I \cap \alpha \subset I'$ . Therefore since

$$N \cap V_\alpha \prec_{\Sigma_2} N,$$

it follows  $I \subset I'$ . Thus  $I = I'$ .

For each  $\xi \in I$  let  $M_\xi \in N$  be such that

(3.1)  $M_\xi \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement}$ ,

(3.2)  $\mathbb{P} \in M_\xi$ ,

(3.3)  $\mathcal{P}(\mathbb{P}) \cap M_\xi = \mathcal{P}(\mathbb{P}) \cap M$ ,

(3.4)  $N \cap V_\xi = M_\xi[G]$ .

By Lemma 27, for each  $\xi_1 < \xi_2$  in  $I$ ,

$$M_{\xi_1} = M_{\xi_2} \cap V_{\xi_1}.$$

Let  $M^* = \cup \{M_\xi \mid \xi \in I\}$ . Thus

(4.1)  $M = M^* \cap V_\alpha$ ,

(4.2)  $N = M^*[G]$ ,

(4.3) for each  $\xi \in I$ ,  $M^* \cap V_\xi \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement}$ .

Finally by (4.3),

$$M^* \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement},$$

since  $I$  is cofinal in  $M^* \cap \text{Ord}$ . ⊢

LEMMA 29. Suppose that  $M \subset N$  are transitive sets,

$$\mathbb{P} \in M$$

and  $G \subset \mathbb{P}$  satisfy:

- (i)  $M \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement}$ ;
- (ii)  $N \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement}$ ;
- (iii)  $G$  is  $M$ -generic;
- (iv)  $N = M[G]$ .

Suppose  $k \geq 1$ ,  $\gamma \in N \cap \text{Ord}$ ,  $\mathbb{P} \in N \cap V_\gamma$ , and that the set

$$\{\xi \in N \cap \text{Ord} \mid N \cap V_\xi \prec_{\Sigma_k} N\}$$

is cofinal in  $N \cap \text{Ord}$ .

Then

$$M \cap V_\gamma \prec_{\Sigma_k} M$$

if and only if  $N \cap V_\gamma \prec_{\Sigma_k} N$ .

PROOF. The case  $k = 1$  is immediate since for  $\gamma > \omega$ ,

$$N \cap V_\gamma \prec_{\Sigma_1} N$$

if and only if  $|N \cap V_\gamma|^N = \gamma$  and

$$M \cap V_\gamma \prec_{\Sigma_1} M$$

if and only if  $|M \cap V_\gamma|^M = \gamma$ .

We now suppose that  $k > 1$ . If

$$M \cap V_\gamma \prec_{\Sigma_k} M$$

then it follows easily that

$$N \cap V_\gamma \prec_{\Sigma_k} N.$$

This is because  $N = M[G]$  and  $\mathbb{P} \in M \cap V_\gamma$ .

Finally suppose that

$$N \cap V_\gamma \prec_{\Sigma_k} N.$$

Let  $I_1^M$  be the set of  $\xi \in M \cap \text{Ord}$  such that

$$M \cap V_\xi \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement}$$

and such that  $\mathbb{P} \in M \cap V_\xi$ . Clearly

$$I_1^M = \{\xi \in M \cap \text{Ord} \mid M \cap V_\xi \prec_{\Sigma_1} M\} \cap \{\xi \in M \cap \text{Ord} \mid \mathbb{P} \in M \cap V_\xi\},$$

and  $I_1^M$  is also the set of  $\xi \in N \cap \text{Ord}$  such that

$$N \cap V_\xi \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement}$$

and such that  $\mathbb{P} \in N \cap V_\xi$ .

By Lemma 27 and Lemma 28, it follows that the set

$$\{(\xi, M \cap V_\xi) \mid \xi \in I_1^M\}$$

is  $\Pi_1$ -definable in  $N$  from the parameter,

$$(\mathbb{P}, G, \mathcal{P}(\mathbb{P}) \cap M).$$

Since

$$M \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement},$$

for each  $\xi \in I_1^M$ ,  $M \cap V_\xi \prec_{\Sigma_1} M$ .

For each  $1 \leq n \leq k$  let  $I_n^M$  be the set of  $\xi \in I_1^M$  such that

$$M \cap V_\xi \prec_{\Sigma_n} M.$$

We prove by induction on  $n$ :

$$(1.1) \quad I_n^M = \{\xi \in I_1^M \mid N \cap V_\xi \prec_{\Sigma_n} N\};$$

$$(1.2) \quad \{(\xi, M \cap V_\xi) \mid \xi \in I_n^M\} \text{ is } \Pi_n\text{-definable in } N \text{ from the parameter,}$$

$$(\mathbb{P}, G, \mathcal{P}(\mathbb{P}) \cap M).$$

We have just proved (1.1)–(1.2) for  $n = 1$ , so we suppose that  $1 < n < k$  and (1.1)–(1.2) hold for  $n$ . We first prove (1.1) holds for  $n + 1$  and for this it suffices to simply show that

$$\{\xi \in I \mid N \cap V_\xi \prec_{\Sigma_{n+1}} N\} \subseteq I_{n+1}^M.$$

Suppose that  $\xi \in I_1^M$  and

$$N \cap V_\xi \prec_{\Sigma_{n+1}} N.$$

We must prove that  $M \cap V_\xi \prec_{\Sigma_{n+1}} M$ . Since  $n < k$  we have that

$$\{\eta \in N \cap \text{Ord} \mid N \cap V_\eta \prec_{\Sigma_n} N\}$$

is cofinal in  $N \cap \text{Ord}$ . Therefore by the induction hypothesis,

$$M \cap V_\xi \prec_{\Sigma_{n+1}} M$$

if and only if

$$(M \cap V_\xi, \{M \cap V_{\xi'} \mid \xi' \in I_n^M \cap \xi\}) \prec_{\Sigma_1} (M, \{M \cap V_{\xi'} \mid \xi' \in I_n^M\}).$$

Since (1.2) holds for  $n$  and since

$$N \cap V_\xi \prec_{\Sigma_{n+1}} N,$$

it follows that

$$(N \cap V_\xi, \{M \cap V_{\xi'} \mid \xi' \in I_n^M \cap \xi\}) \prec_{\Sigma_1} (N, \{M \cap V_{\xi'} \mid \xi' \in I_n^M\}),$$

and so  $M \cap V_\xi \prec_{\Sigma_{n+1}} M$ .

Finally we prove that (1.2) holds for  $n + 1$ . Note that  $M \cap V_\xi \prec_{\Sigma_{n+1}} M$  if and only if for all  $a \in M \cap V_\xi$  and for all formulas  $\phi(x)$  if

$$\{(\xi', M \cap V_{\xi'}) \mid \xi' \in I_n^M \text{ and } M \cap V_{\xi'} \models \phi[a]\} \neq \emptyset$$

then

$$\{(\xi', M \cap V_{\xi'}) \mid \xi' \in I_n^M \text{ and } M \cap V_{\xi'} \models \phi[a]\} \cap V_\xi \neq \emptyset.$$

This implies that (1.2) holds for  $n + 1$  since (1.2) holds for  $n$ .

Thus (1.1) and (1.2) hold for all  $n \leq k$  and in particular (1.1) holds for  $n = k$ . Therefore

$$M \cap V_\gamma \prec_{\Sigma_k} M,$$

and this completes the proof of the lemma.  $\dashv$

LEMMA 30. *Suppose that  $N_1$  is a countable transitive set,*

$$N_1 \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement},$$

*and that  $k \geq 1$ . Let*

$$I = \{\gamma \in N_1 \cap \text{Ord} \mid N_1 \cap V_\gamma \prec_{\Sigma_k} N_1\}$$

*and suppose that  $I$  is cofinal in  $N_1 \cap \text{Ord}$ .*

- (1) *Suppose  $\gamma \in I$ . Then for each  $N \in \mathbb{V}_{N_1 \cap V_\gamma}$  there exists  $N^* \in \mathbb{V}_{N_1}$  such that  $N = N^* \cap V_\gamma$  and*

$$N \prec_{\Sigma_k} N^*.$$

- (2) *Suppose that  $N \in \mathbb{V}_{N_1}$ . Then there exists  $\xi \in N_1 \cap \text{Ord}$  such that for all  $\gamma \in I \setminus \xi$ ,*

$$N \cap V_\gamma \in \mathbb{V}_{N \cap V_\gamma}$$

$$\text{and } N \cap V_\gamma \prec_{\Sigma_k} N.$$

PROOF. We first prove (1). Fix  $\gamma \in I$ , let  $N_2 = N_1 \cap V_\gamma$  and fix  $N \in \mathbb{V}_{N_2}$ . Then there exists a finite sequence,

$$\langle (\mathbb{P}_i, G_i, M_i) : i \leq m \rangle$$

such that

- (1.1)  $M_0 = N_2$  and  $M_m = N$ ,
- (1.2) for all  $i < m$ ,  $M_i \in \mathbb{V}_{N_2}$ ,
- (1.3) for all  $i + 1 \leq m$  either
  - a)  $\mathbb{P}_i \in M_i$ ,  $G_i \subset \mathbb{P}_i$  is  $M_i$ -generic and  $M_{i+1} = M_i[G_i]$ , or
  - b)  $\mathbb{P}_i \in M_{i+1}$ ,  $G_i \subset \mathbb{P}_i$  is  $M_{i+1}$ -generic and  $M_i = M_{i+1}[G_i]$ .

We prove by induction on  $i \leq m$  that there exists  $M_i^* \in \mathbb{V}_{N_1}$  such that

- (2.1)  $M_i^* \cap V_\gamma = M_i$ ,
- (2.2)  $M_i \prec_{\Sigma_k} M_i^*$ .

If  $i = 0$  then  $M_0^* = N_1$  and (2.1)–(2.2) are immediate.

Suppose  $M_i^*$  exists satisfying (2.1)–(2.2) and that  $i + 1 \leq m$ . There are two cases. The first case is that  $M_{i+1} = M_i[G_i]$ . Then set  $M_{i+1}^* = M_i^*[G_i]$ . Thus  $M_{i+1}^* \in \mathbb{V}_{N_1}$  since  $M_i^* \in \mathbb{V}_{N_1}$  and by Lemma 29

$$M_{i+1} \prec_{\Sigma_k} M_{i+1}^*.$$

The second case is that  $M_i = M_{i+1}[G_i]$ . By Lemma 28 there exists  $M_{i+1}^*$  such that  $M_i^* = M_{i+1}^*[G_i]$  and such that  $M_{i+1} = M_{i+1}^* \cap V_\gamma$ . By Lemma 29 and the induction hypothesis,

$$M_{i+1} \prec_{\Sigma_k} M_{i+1}^*.$$

This proves that for all  $i \leq m$ ,  $M_i^*$  exists satisfying (2.1)–(2.2). Set  $N^* = M_m^*$ . This proves (1). The proof of (2) is similar.  $\dashv$

LEMMA 31. *Suppose  $M$  is a countable transitive set,*

$$M \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement},$$

*$\gamma \in M \cap \text{Ord}$  and that*

$$M_\gamma \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement}.$$

*Suppose  $G \subset \text{Coll}(\omega, \gamma)$  is  $M$ -generic. Then for each sentence  $\phi$  the following are equivalent.*

- (1) *For all  $N \in \mathbb{V}_{M \cap V_\gamma}$ ,  $N \models \phi$ .*
- (2)  *$M[G] \models$  “For all  $N \in \mathbb{V}_{M \cap V_\gamma}$ ,  $N \models \phi$ ”.*

PROOF. By absoluteness,

$$M[G] \cap V_{\omega+1} \prec_{\Sigma_1} V_{\omega+1}$$

and

$$\mathbb{V}_{M \cap V_\gamma} \cap M[G] = (\mathbb{V}_{M \cap V_\gamma})^{M[G]}.$$

The lemma follows.  $\dashv$

Suppose that  $\phi$  is a sentence and let  $n$  be the length of  $\phi$ . Let  $\phi^*$  be a sentence which expresses:

*Suppose  $\gamma$  is an ordinal such that*

$$V_\gamma \prec_{\Sigma_{n+1}} V$$

*and that  $X \prec V_\gamma$  is a countable elementary substructure. Let  $M_X$  be the transitive collapse of  $X$ . Then*

$$N \models \phi$$

*for each  $N \in \mathbb{V}_{M_X}$ .*

The next lemma shows that  $\phi^*$  is as required.

LEMMA 32. *Suppose that  $M$  is a countable transitive set such that*

$$M \models \text{ZFC}.$$

*Then the following are equivalent.*

- (1)  *$M \models \phi^*$ .*
- (2) *For each  $N \in \mathbb{V}_M$ ,  $N \models \phi$ .*

PROOF. Let  $\gamma$  be an ordinal of  $M$  such that

$$M \cap V_\gamma \prec_{\Sigma_{n+1}} M$$

and let

$$I = \{\xi \in M \cap \text{Ord} \mid M \cap V_\xi \prec_{\Sigma_n} M\}.$$

We first show that (1) implies (2). Suppose  $\xi \in I \cap \gamma$ . We claim that for all  $N \in \mathbb{V}_{M \cap V_\xi}$ ,  $N \models \phi$ . To see this let

$$X \prec M \cap V_\gamma$$

be an elementary substructure such that  $\xi \in X$ ,  $X \in M$ , and such that  $|X|^M = \omega$ . Let  $M_X$  be the transitive collapse of  $X$  and let  $\xi_X$  be the image of  $\xi$  under the collapsing map. Thus

$$M_X \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement},$$

and  $M_X \cap V_{\xi_X} \prec_{\Sigma_n} M_X$ .

By (1),  $M \models \phi^*$ . Therefore by absoluteness and since

$$(\mathbb{V}_{M_X})^M = \mathbb{V}_{M_X} \cap M,$$

for all  $N \in \mathbb{V}_{M_X}$ ,  $N \models \phi$ . Therefore if  $G \subset \text{Coll}(\omega, \xi)$  is  $M_X$ -generic then for all  $N \in \mathbb{V}_{M_X} \cap M_X[G]$ ,  $N \models \phi$ . Since  $X \prec M \cap V_\gamma$  it follows that if  $G \subset \text{Coll}(\omega, \xi)$  is  $M$ -generic then for all  $N \in \mathbb{V}_{M \cap V_\xi} \cap M[G]$ ,  $N \models \phi$  and so by Lemma 31, for all  $N \in \mathbb{V}_{M \cap V_\xi}$ ,  $N \models \phi$ .

In summary we have proved that for each  $\xi \in I \cap \gamma$ , for all  $N \in \mathbb{V}_{M \cap V_\xi}$ ,  $N \models \phi$ . By induction on  $k$  it follows that for each  $k \geq 1$ , the

$$\{M \cap V_\xi \mid M \cap V_\xi \prec_{\Sigma_k} M\}$$

is  $\Pi_k$ -definable in  $M$ . Therefore by Lemma 31, and since

$$M_\gamma \prec_{\Sigma_{n+1}} M,$$

it follows that for each  $\xi \in I$ , for all  $N \in \mathbb{V}_{M \cap V_\xi}$ ,  $N \models \phi$ .

Finally suppose that  $N \in \mathbb{V}_M$ . Then by Lemma 30, there exists  $\xi \in I$  such that  $N \cap V_\xi \in \mathbb{V}_{M \cap V_\xi}$  and such that

$$N \cap V_\xi \prec_{\Sigma_n} N.$$

Since  $\phi$  has length at most  $n$ ,  $N \models \phi$  if and only if

$$N \cap V_\xi \models \phi.$$

Therefore  $N \models \phi$ . This proves (2).

We now suppose (2) holds and prove (1). Let

$$X \prec M \cap V_\gamma$$

be an elementary substructure such that  $X \in M$  and such that  $|X|^M = \omega$ . Let  $M_X$  be the transitive collapse of  $X$ . We must show that for all  $N \in \mathbb{V}_{M_X} \cap M$ ,  $N \models \phi$ .

Let

$$I_X = \{\xi \in M_X \cap \text{Ord} \mid M_X \cap V_\xi \prec_{\Sigma_n} M_X\}.$$

Thus  $I_X$  is the image of  $I \cap X$  under the collapsing map.

By (2) and Lemma 30(1), for each  $\xi \in I$ , for each  $N \in \mathbb{V}_{M \cap V_\xi}$ ,  $N \models \phi$ . Therefore by Lemma 31, for each  $\xi \in I_X$ , for each  $N \in \mathbb{V}_{M_X \cap V_\xi}$ ,  $N \models \phi$ .

Suppose  $N \in \mathbb{V}_{M_X} \cap M$ . Then by Lemma 30(2), there exists  $\gamma_0 \in I_X$  such that  $N \cap V_{\gamma_0} \in \mathbb{V}_{M_X \cap V_{\gamma_0}}$  and such that

$$N \cap V_{\gamma_0} \prec_{\Sigma_n} N.$$

Since  $N \cap V_{\gamma_0} \in \mathbb{V}_{M_X \cap V_{\gamma_0}}$ ,  $N \cap V_{\gamma_0} \models \phi$  and therefore since  $N \cap V_{\gamma_0} \prec_{\Sigma_n} N$ ,  $N \models \phi$ .

Therefore, for all  $N \in \mathbb{V}_{M_X} \cap M$ ,  $N \models \phi$ . This proves (1).  $\dashv$

I finish by discussing the generalization of Lemma 32 to any restricted generic-multiverse of the following form.

Suppose  $\psi(x)$  is a formula in the language of set theory and that  $M$  is a countable transitive set such that

$$M \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement}.$$

Let  $\mathbb{V}_M^{(\psi)} \subseteq \mathbb{V}_M$  be the be smallest set such that  $M \in \mathbb{V}_M^{(\psi)}$  and such that for all pairs,  $(M_1, M_2)$ , of countable transitive sets if

- (1)  $M_1 \models \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement}$ ,
- (2)  $M_2 = M_1[G]$  for some set  $G$  such that  $G$  is  $M_1$ -generic for some partial order  $\mathbb{P}$  such that  $M_1 \models \psi[\mathbb{P}]$ ,
- (3) either  $M_1 \in \mathbb{V}_M^{(\psi)}$  or  $M_2 \in \mathbb{V}_M^{(\psi)}$ ,

then both  $M_1$  and  $M_2$  are in  $\mathbb{V}_M^{(\psi)}$ .

For example one can choose  $\psi$  such that

$$\mathbb{V}_M^{(\psi)} = \{N \in \mathbb{V}_M \mid (\omega_1)^M = (\omega_1)^N\}$$

or such that

$$\mathbb{V}_M^{(\psi)} = \{N \in \mathbb{V}_M \mid (\text{cardinals})^M = (\text{cardinals})^N\}.$$

Suppose that  $\phi$  is a sentence and let  $n = \text{length}(\phi) + \text{length}(\psi)$ . Let  $\phi^{(\psi)}$  be a sentence which expresses:

*Suppose  $\gamma$  is an ordinal such that*

$$V_\gamma \prec_{\Sigma_{n+1}} V$$

*and that  $X \prec V_\gamma$  is a countable elementary substructure. Let  $M_X$  be the transitive collapse of  $X$ . Then*

$$N \models \phi$$

*for each  $N \in \mathbb{V}_{M_X}^{(\psi)}$ .*

The proof of Lemma 32 easily adapts to prove the following generalization.

**LEMMA 33.** *Suppose that  $M$  is a countable transitive set such that*

$$M \models \text{ZFC}.$$

*Then the following are equivalent.*

- (1)  $M \models \phi^{(\psi)}$ .
- (2) *For each  $N \in \mathbb{V}_M^{(\psi)}$ ,  $N \models \phi$ .*

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# $\omega$ -MODELS OF FINITE SET THEORY

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**Abstract.** Finite set theory, here denoted  $\text{ZF}_{\text{fin}}$ , is the theory obtained by replacing the axiom of infinity by its negation in the usual axiomatization of ZF (Zermelo-Fraenkel set theory). An  $\omega$ -model of  $\text{ZF}_{\text{fin}}$  is a model in which every set has at most finitely many elements (as viewed externally). Mancini and Zambella (2001) employed the Bernays-Rieger method of permutations to construct a recursive  $\omega$ -model of  $\text{ZF}_{\text{fin}}$  that is nonstandard (i.e., not isomorphic to the hereditarily finite sets  $V_\omega$ ). In this paper we initiate the metamathematical investigation of  $\omega$ -models of  $\text{ZF}_{\text{fin}}$ . In particular, we present a new method for building  $\omega$ -models of  $\text{ZF}_{\text{fin}}$  that leads to a perspicuous construction of recursive nonstandard  $\omega$ -models of  $\text{ZF}_{\text{fin}}$  without the use of permutations. Furthermore, we show that every recursive model of  $\text{ZF}_{\text{fin}}$  is an  $\omega$ -model. The central theorem of the paper is the following:

**THEOREM A.** *For every graph  $(A, F)$ , where  $F$  is a set of unordered pairs of  $A$ , there is an  $\omega$ -model  $\mathfrak{M}$  of  $\text{ZF}_{\text{fin}}$  whose universe contains  $A$  and which satisfies the following conditions:*

- (1)  *$(A, F)$  is definable in  $\mathfrak{M}$ ;*
- (2) *Every element of  $\mathfrak{M}$  is definable in  $(\mathfrak{M}, a)_{a \in A}$ ;*
- (3) *If  $(A, F)$  is pointwise definable, then so is  $\mathfrak{M}$ ;*
- (4)  *$\text{Aut}(\mathfrak{M}) \cong \text{Aut}(A, F)$ .*

Theorem A enables us to build a variety of  $\omega$ -models with special features, in particular:

**COROLLARY 1.** *Every group can be realized as the automorphism group of an  $\omega$ -model of  $\text{ZF}_{\text{fin}}$ .*

**COROLLARY 2.** *For each infinite cardinal  $\kappa$  there are  $2^\kappa$  rigid nonisomorphic  $\omega$ -models of  $\text{ZF}_{\text{fin}}$  of cardinality  $\kappa$ .*

**COROLLARY 3.** *There are continuum-many nonisomorphic pointwise definable  $\omega$ -models of  $\text{ZF}_{\text{fin}}$ .*

We also establish that PA (Peano arithmetic) and  $\text{ZF}_{\text{fin}}$  are not bi-interpretable by showing that they differ even for a much coarser notion of equivalence, to wit *sentential equivalence*.

**§1. Introduction.** In 1953, Kreisel [Kr] and Mostowski [Mos] independently showed that certain finitely axiomatizable systems of set theory formulated in an *expansion* of the usual language  $\{\in\}$  of set theory do not possess any recursive models. This result was improved in 1958 by Rabin [Ra] who found a “familiar” finitely axiomatizable first order theory that has no recursive model: Gödel-Bernays<sup>1</sup> set theory GB without the axiom of infinity (note

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<sup>1</sup>We have followed Mostowski’s lead in our adoption of the appellation GB, but some set theory texts refer to this theory as BG. To make matters more confusing, the same theory is also known in the literature as VNB (*von Neumann-Bernays*) and NBG (*von Neumann-Bernays-Gödel*).

that GB can be formulated in the language  $\{\in\}$  with no extra symbols). These discoveries were overshadowed by Tennenbaum's celebrated 1959 theorem that characterizes the standard model of PA (Peano arithmetic) as the only recursive model of PA up to isomorphism, thereby shifting the focus of the investigation of the complexity of models from set theory to arithmetic. We have come a long way since Tennenbaum's pioneering work in understanding the contours of the "Tennenbaum boundary" that separates those fragments of PA that have a recursive nonstandard model (such as  $IOpen$ ) from those which do not (such as  $I\exists_1$ ), but the study of the complexity of models of arithmetic and its fragments remains a vibrant research area with many intriguing open questions.<sup>2</sup>

The point of departure for the work presented here is Mancini and Zambella's 2001 paper [MZ] that focuses on *Tennenbaum phenomena in set theory*. Mancini and Zambella introduced a weak fragment (dubbed  $KP\Sigma_1$ )<sup>3</sup> of Kripke-Platek set theory KP, and showed that the only recursive model of  $KP\Sigma_1$  up to isomorphism is the standard one, i.e.,  $(V_\omega, \in)$ , where  $V_\omega$  is the set of hereditarily finite sets. In contrast, they used the *Bernays-Rieger*<sup>4</sup> permutation method to show that the theory  $ZF_{fin}$  obtained by replacing the axiom of infinity by its negation in the usual axiomatization of ZF (Zermelo-Fraenkel set theory) has a recursive nonstandard model. The Mancini-Zambella recursive nonstandard model also has the curious feature of being an  $\omega$ -model in the sense that every element of the model, as viewed externally, has at most finitely many members, a feature that caught our imagination and prompted us to initiate the systematic investigation of the metamathematics of  $\omega$ -models of  $ZF_{fin}$ .

The plan of the paper is as follows. Preliminaries are dealt with in Section 2, in which we review key definitions, establish notation, and discuss a host of background results. In Section 3 we present a simple robust construction of  $\omega$ -models of  $ZF_{fin}$  (Theorem 3.4), a construction that, among other things, leads to a perspicuous proof of the existence of  $\omega$ -models of  $ZF_{fin}$  in every infinite cardinality (Corollary 3.7), and the existence of infinitely many non-isomorphic recursive nonstandard  $\omega$ -models of  $ZF_{fin}$  without the use of permutations (Corollary 3.9, Remark 3.10(b)). In the same section, we also show

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<sup>2</sup>See, e.g., the papers by Kaye [Ka] and Schmerl [Sch-2] in this volume, and Mohsenipour's [Moh].

<sup>3</sup>The axioms of  $KP\Sigma_1$  consist of Extensionality, Pairs, Union, Foundation,  $\Delta_0$ -Comprehension,  $\Delta_0$ -Collection, and the scheme of  $\in$ -Induction (defined in part (f) of Remark 2.2) only for  $\Sigma_1$  formulas. Note that  $KP\Sigma_1$  does not include the axiom of infinity.

<sup>4</sup>In [MZ] this method is incorrectly referred to as the *Fraenkel-Mostowski* permutation method, but this method was invented by Bernays (announced in [Be-1, p. 9], and presented in [Be-2]) and fine-tuned by Rieger [Ri] in order to build models of set theory that violate the regularity (foundation) axiom, e.g., by containing sets  $x$  such that  $x = \{x\}$ . In contrast, the Fraenkel-Mostowski method is used to construct models of set theory with atoms in which the axiom of choice fails.

that  $\text{ZF}_{\text{fin}}$  is not completely immune to Tennenbaum phenomena by demonstrating that every recursive model of  $\text{ZF}_{\text{fin}}$  is an  $\omega$ -model (Theorem 3.11). In Section 4 we fine-tune the method of Section 3 to show the existence of a wealth of nonisomorphic  $\omega$ -models of  $\text{ZF}_{\text{fin}}$  with special features. The central theorem of Section 4 is Theorem 4.2 which shows that every graph can be canonically coded into an  $\omega$ -model of  $\text{ZF}_{\text{fin}}$ . Coupled with classical results in graph theory, this result yields many corollaries. For example, every group (of any cardinality) can be realized as the automorphism group of an  $\omega$ -model of  $\text{ZF}_{\text{fin}}$  (Corollary 4.4), and there are continuum-many nonisomorphic pointwise definable  $\omega$ -models of  $\text{ZF}_{\text{fin}}$  (Corollary 4.7). In Section 5 we establish that PA (Peano arithmetic) and  $\text{ZF}_{\text{fin}}$  are not bi-interpretable by showing that they differ even for a much coarser notion of equivalence, to wit *sentential equivalence* (Theorem 5.1). This complements the work of Kaye-Wong<sup>5</sup> [KW] on the definitional equivalence (or synonymy) of PA and  $\text{ZF}_{\text{fin}}$  + “every set has a transitive closure”. We close the paper with Section 6, in which we present open questions and concluding remarks.

We also wish to take this opportunity to thank the anonymous referee for helpful comments and corrections, and Robert Solovay for catching a blooper in an earlier draft of this paper.

**§2. Preliminaries.** In this section we recall some key definitions, establish notation, and review known results.

DEFINITIONS/OBSERVATIONS 2.1. (a) Models of set theory are *directed graphs*<sup>6</sup> (hereafter: *digraphs*), i.e., structures of the form  $\mathfrak{M} = (M, E)$ , where  $E$  is a binary relation on  $M$  that interprets  $\in$ . We often write  $xEy$  as a shorthand for  $\langle x, y \rangle \in E$ . For  $c \in M$ ,  $c_E$  is the set of “elements” of  $c$ , i.e.,

$$c_E := \{m \in M : mEc\}.$$

$\mathfrak{M}$  is *nonstandard* if  $E$  is not well-founded, i.e., if there is a sequence  $\langle c_n : n \in \omega \rangle$  of elements of  $M$  such that  $c_{n+1}Ec_n$  for all  $n \in \omega$ .

(b) We adopt the terminology of Baratella and Ferro [BF] of using EST (elementary set theory) to refer to the following theory of sets

EST := Extensionality + Empty Set + Pairs + Union + Replacement.

(c) The theory  $\text{ZF}_{\text{fin}}$  is obtained by replacing the axiom of infinity by its negation in the usual axiomatization of ZF (Zermelo-Fraenkel set theory). More explicitly:

$$\text{ZF}_{\text{fin}} := \text{EST} + \text{Power set} + \text{Regularity}^7 + \neg\text{Infinity}.$$

<sup>5</sup>See Remark 2.2(f) for more detail concerning the Kaye-Wong paper.

<sup>6</sup>We will also have ample occasion to deal with (undirected) graphs, i.e., structures of the form  $(A, F)$ , where  $F$  is a set of *unordered* pairs from  $A$ .

Here Infinity is the usual axiom of infinity, i.e.,

$$\text{Infinity} := \exists x \left( \emptyset \in x \wedge \forall y (y \in x \rightarrow y^+ \in x) \right),$$

where  $y^+ := y \cup \{y\}$ .

(d)  $\text{Tran}(x)$  is the first order formula that expresses the statement “ $x$  is transitive”, i.e.,

$$\text{Tran}(x) := \forall y \forall z (z \in y \in x \rightarrow z \in x).$$

(e)  $\text{TC}(x)$  is the first order formula that expresses the statement “the transitive closure of  $x$  is a set”, i.e.,

$$\text{TC}(x) := \exists y (x \subseteq y \wedge \text{Tran}(y)).$$

Overtly, the above formula just says that some superset of  $x$  is transitive, but it is easy to see that  $\text{TC}(x)$  is equivalent within EST to the following statement expressing “there is a smallest transitive set that contains  $x$ ”

$$\exists y (x \subseteq y \wedge \text{Tran}(y) \wedge \forall z ((x \subseteq z \wedge \text{Tran}(z)) \rightarrow y \subseteq z)).$$

(f) TC denotes the *transitive closure* axiom

$$\text{TC} := \forall x \text{TC}(x).$$

Let  $V_\omega$  be the set of hereditarily finite sets. It is easy to see that  $\text{ZF}_{\text{fin}} + \text{TC}$  holds in  $V_\omega$ . However, it has long been known<sup>8</sup> that  $\text{ZF}_{\text{fin}} \not\models \text{TC}$ .

(g)  $\mathbb{N}(x)$  [read as “ $x$  is a natural number”] is the formula

$$\text{Ord}(x) \wedge \forall y \in x^+ (y \neq \emptyset \rightarrow \exists z (\text{Ord}(z) \wedge y = z^+)),$$

where  $\text{Ord}(x)$  expresses “ $x$  is a (von Neumann) ordinal”, i.e., “ $x$  is a transitive set that is well-ordered by  $\in$ ”. It is well-known that with this interpretation, the full induction scheme  $\text{Ind}_{\mathbb{N}}$ , consisting of the universal closure of formulas of the following form is provable within EST:

$$(\theta(0) \wedge \forall x (\mathbb{N}(x) \wedge \theta(x) \rightarrow \theta(x^+))) \rightarrow \forall x (\mathbb{N}(x) \rightarrow \theta(x)).$$

Note that  $\theta$  is allowed to have suppressed parameters, and these parameters are not required to lie in  $\mathbb{N}$ . Coupled with the fact that  $\text{ZF}_{\text{fin}}$  is a sequential theory<sup>9</sup>, this shows that for each positive integer  $n$ , there is a formula  $\text{Tr}_n(x)$  such that, provably in  $\text{ZF}_{\text{fin}}$ ,  $\text{Tr}_n(x)$  is a truth-predicate for the class of formulas  $\mathcal{Q}_n$  with

<sup>7</sup>The regularity axiom is also known as the *foundation* axiom, stating that every nonempty set has an  $\in$ -minimal element.

<sup>8</sup>Hájek-Vopěnka [HV] showed that TC is not provable in the theory  $\text{GB}_{\text{fin}}$ , which is obtained from GB (Gödel-Bernays theory of classes) by replacing the axiom of infinity by its negation. Later Hauschild [Ha] gave a direct construction of a model of  $\text{ZF}_{\text{fin}} + \neg \text{TC}$ .

<sup>9</sup>Sequential theories are those that are equipped with a “ $\beta$ -function” for coding sequences. More specifically, using  $\langle x, y \rangle$  for the usual Kuratowski ordered pair of  $x$  and  $y$ , the function  $\beta(x, y)$  defined via  $\beta(x, y) = z$  iff  $\langle x, z \rangle \in y$ , conveniently serves in EST as a  $\beta$ -function (when  $y$  is restricted to “functional” sets, i.e.,  $y$  should contain at most one ordered pair  $\langle x, z \rangle$  for a given  $x$ ).

at most  $n$  alternations of quantifiers. Coupled with the fact that any formula in  $\mathcal{Q}_n$  that is provable in predicate logic has a proof all of whose components lie in  $\mathcal{Q}_n$  (which follows from Herbrand's Theorem [HP, Thm. 3.30, Ch. III]), this shows that  $\text{ZF}_{\text{fin}}$  is *essentially reflexive*, i.e., any consistent extension of  $\text{ZF}_{\text{fin}}$  proves the consistency of each of its finite subtheories. Therefore  $\text{ZF}_{\text{fin}}$  is not finitely axiomatizable.

(h) For a model  $\mathfrak{M} \models \text{EST}$ , and  $x \in M$ , we say that  $x$  is  $\mathbb{N}$ -finite if there is a bijection in  $\mathfrak{M}$  between  $x$  and some element of  $\mathbb{N}^{\mathfrak{M}}$ . Let:

$$(V_{\omega})^{\mathfrak{M}} := \{m \in M : \mathfrak{M} \models \text{"TC}(m) \text{ and } m \text{ is } \mathbb{N}\text{-finite"}\}$$

It is easy to see that

$$(V_{\omega})^{\mathfrak{M}} \models \text{ZF}_{\text{fin}} + \text{TC}.$$

This provides an interpretation of the theory  $\text{ZF}_{\text{fin}} + \text{TC}$  within EST. On the other hand, the existence of recursive nonstandard models of  $\text{ZF}_{\text{fin}}$  shows that, conversely,  $\text{ZF}_{\text{fin}} + \neg \text{TC}$  is also interpretable in EST. One can show that the above interpretation of  $\text{ZF}_{\text{fin}} + \text{TC}$  within EST is *faithful* (i.e., the sentences that are provably true in the interpretation are precisely the logical consequences of  $\text{ZF}_{\text{fin}} + \text{TC}$ ), but that the Mancini-Zambella interpretation [MZ, Theorem 3.1] of  $\text{ZF}_{\text{fin}} + \neg \text{TC}$  within EST, is not faithful<sup>10</sup>.

(i)  $\tau(n, x)$  is the term expressing “the  $n$ -th approximation to the transitive closure of  $\{x\}$  (where  $n$  is a natural number)”. Informally speaking,

- $\tau(0, x) = \{x\}$ ;
- $\tau(n+1, x) = \tau(n, x) \cup \{y : \exists z (y \in z \in \tau(n, x))\}$ .

Thanks to the coding apparatus of EST for dealing with finite sequences, and the provability of  $\text{Ind}_{\mathbb{N}}$  within EST (both mentioned earlier in part (g)), the above informal recursion can be formalized within EST to show that

$$\text{EST} \vdash \forall n \forall x (\mathbb{N}(n) \rightarrow \exists! y (\tau(n, x) = y)).$$

This leads to the following important observation:

(j) Even though the transitive closure of a set need not form a set in EST (or even in  $\text{ZF}_{\text{fin}}$ ), for an  $\omega$ -model  $\mathfrak{M}$  the transitive closure  $\tau(c)$  of  $\{c\}$  is *first order definable* via:

$$\tau^{\mathfrak{M}}(c) := \{m \in M : \mathfrak{M} \models \exists n (\mathbb{N}(n) \wedge m \in \tau(n, c))\}.$$

This shows that, in the worst case scenario, transitive closures behave like proper classes in  $\omega$ -models of  $\mathfrak{M}$ . Note that if the set  $\mathbb{N}^{\mathfrak{M}}$  of natural numbers

<sup>10</sup>Because the interpretation provably satisfies the sentence “the universe can be built from the transitive closure of an element whose transitive closure forms an  $\omega^*$ -chain”, a sentence that is not a theorem of  $\text{ZF}_{\text{fin}} + \neg \text{TC}$ .

of  $\mathfrak{M}$  contains nonstandard elements, then the *external* transitive closure

$$\bigcup_{n \in \omega} \tau^{\mathfrak{M}}(n, c)$$

might be a proper subset of the  $\mathfrak{M}$ -transitive closure  $\tau^{\mathfrak{M}}(c)$ . However, if  $\mathfrak{M}$  is an  $\omega$ -model, then the  $\mathfrak{M}$ -transitive closure of  $\{c\}$  coincides with the external transitive closure of  $\{c\}$ .

**REMARK 2.2.** (a) Vopěnka [Vo-2] has shown that  $\text{ZF}_{\text{fin}} \setminus \{\text{Regularity}\}$  is provable from the fragment (dubbed VF in [BF]) of AST (Alternative Set Theory), whose axioms consist of Extensionality, Empty set, Adjunction (given sets  $x$  and  $y$ , we have that  $x \cup \{y\}$  exists), and the schema of Set-induction, consisting of the universal closure of formulas of the form ( $\theta$  is allowed to have suppressed parameters)

$$(\theta(0) \wedge \forall x \forall y (\theta(x) \rightarrow \theta(x \cup \{y\}))) \rightarrow \forall x \theta(x).$$

(b) Besides  $\neg$ Infinity there are at least two other noteworthy first order statements that can be used to express “every set is finite”:

- $\text{Fin}_{\mathbb{N}}$ : Every set is  $\mathbb{N}$ -finite.
- $\text{Fin}_D$ : Every set is Dedekind-finite, i.e., no set is equinumerous to a proper subset of itself.

It is easy to see that EST proves  $\text{Fin}_{\mathbb{N}} \rightarrow \text{Fin}_D \rightarrow \neg$ Infinity. By a theorem of Vopěnka [Vo-1], Power set and the well-ordering theorem (and therefore the axiom of choice) are provable within  $\text{EST} + \text{Fin}_{\mathbb{N}}$  (see [BF, Theorem 5] for an exposition).

(c) Kunen [BF, Sec. 7] has shown the consistency of the theory  $\text{EST} + \neg$ Infinity +  $\neg \text{Fin}_{\mathbb{N}}$  using the Fraenkel-Mostowski permutations method.

(d) In contrast with Kunen’s aforementioned result,  $\text{Fin}_{\mathbb{N}}$  is provable within  $\text{ZF}_{\text{fin}} \setminus \{\text{Regularity}\}$  (i.e., within  $\text{EST} + \text{Powerset} + \neg$ Infinity). To see this, work in  $\text{ZF}_{\text{fin}} \setminus \{\text{Regularity}\}$  and suppose to the contrary that there is an element  $x$  that is not  $\mathbb{N}$ -finite. By an easy induction, we find that for every natural number  $n$  there is a subset  $y$  of  $x$  that is equinumerous with  $n$ . Now we define the function  $F$  on the powerset of  $x$  by:

$$F(y) := n, \text{ if } n \text{ is a natural number that is equinumerous with } y; \\ \text{otherwise } F(y) = 0.$$

It is easy to see that the range of  $F$  is  $\omega$ . Hence by Replacement,  $\omega$  is a set. Quod non.

(e) As mentioned earlier in part (a),  $\text{VF} \vdash \text{ZF}_{\text{fin}} \setminus \{\text{Regularity}\}$ . The provability of both  $\text{Fin}_{\mathbb{N}}$  and  $\text{Ind}_{\mathbb{N}}$  in  $\text{ZF}_{\text{fin}}$  can be used to show that  $\text{VF} + \text{Regularity}$  and  $\text{EST} + \text{Fin}_{\mathbb{N}} + \text{Regularity}$  axiomatize the same first order theory as  $\text{ZF}_{\text{fin}}$ .

(f) As observed by Kaye and Wong [KW, Prop. 12] within EST, the principle TC is equivalent to the scheme of  $\in$ -Induction consisting of statements of the

following form ( $\theta$  is allowed to have suppressed parameters)

$$\forall y (\forall x \in y \theta(x) \rightarrow \theta(y)) \rightarrow \forall z \theta(z).$$

In the same paper Kaye and Wong showed the following strong form of bi-interpretability<sup>11</sup> between PA and  $ZF_{\text{fin}} + TC$ , known as *definitional equivalence* (or *synonymity*, in the sense of [Vi-2, Sec. 4.8.2]) by showing that:

- (1) TC holds in the Ackermann interpretation Ack of  $ZF_{\text{fin}}$  within PA, i.e.,  $\text{Ack} : ZF_{\text{fin}} + TC \rightarrow \text{PA}$ ; and
- (2) There is an interpretation  $B : \text{PA} \rightarrow ZF_{\text{fin}} + TC$  such that  $\text{Ack} \circ B = \text{id}_{\text{PA}}$  and  $B \circ \text{Ack} = \text{id}_{ZF_{\text{fin}} + TC}$ .

The above result suggests that, contrary to a popular misconception, PA might *not* be bi-interpretable with  $ZF_{\text{fin}}$  alone. Indeed, we shall establish a strong form of the failure of the bi-interpretability between PA and  $ZF_{\text{fin}}$  in Theorem 5.1. Note that, in contrast, by a very general result<sup>12</sup> in interpretability theory, PA and  $ZF_{\text{fin}}$  are mutually faithfully interpretable.

### §3. Building $\omega$ -models.

**DEFINITION 3.1.** Suppose  $\mathfrak{M}$  is a model of EST.  $\mathfrak{M}$  is an  $\omega$ -model if  $|x_E|$  is finite for every  $x \in M$  satisfying  $\mathfrak{M} \models \text{“}x \text{ is } \mathbb{N}\text{-finite”}$ .

It is easy to see that  $\mathfrak{M}$  is an  $\omega$ -model iff  $(\mathbb{N}, \in)^{\mathfrak{M}}$  is isomorphic to the standard natural numbers.<sup>13</sup> This observation can be used to show that  $\omega$ -models of  $ZF_{\text{fin}}$  are precisely the models of the second order theory  $ZF_{\text{fin}}^2$  obtained from  $ZF_{\text{fin}}$  by replacing the replacement scheme by its second order analogue.

The following proposition provides a useful *graph-theoretic* characterization of  $\omega$ -models of  $ZF_{\text{fin}}$ . Note that even though  $ZF_{\text{fin}}$  is not finitely axiomatizable,<sup>14</sup> the equivalence of (a) and (b) of Proposition 3.2 shows that there is a single sentence in the language of set theory whose  $\omega$ -models are precisely  $\omega$ -models of  $ZF_{\text{fin}}$ .

- Recall that a vertex  $x$  of a digraph  $G := (X, E)$  has *finite in-degree* if  $x_E$  is finite; and  $G$  is *acyclic* if there is no finite sequence  $x_1 E x_2 \cdots E x_{n-1} E x_n$  in  $G$  with  $x_1 = x_n$ .

<sup>11</sup>See the paragraph preceding Theorem 3.12 for more detail on bi-interpretability.

<sup>12</sup>See [Vi-1, Lemma 5.4] for the precise formulation of this result in a general setting, which shows that any  $\Sigma_1^0$ -sound theory can be faithfully interpreted in a sufficiently strong theory. Visser’s result extends earlier work of Lindström [Li, Ch. 6, Sec. 2, Thm. 13] which dealt with a similar phenomenon in the specific confines of theories extending PA.

<sup>13</sup>Note that there are really two salient notions of  $\omega$ -model, to wit the notion we defined here could be called  $\omega_{\mathbb{N}}$ -model, and the notion defined using ‘Dedekind-finite’ instead of ‘ $\mathbb{N}$ -finite’ may be called  $\omega_D$ -model. For our study of  $ZF_{\text{fin}}$  the choice is immaterial, since  $ZF_{\text{fin}}$  proves that every set is  $\mathbb{N}$ -finite (see part (d) of Remark 2.2).

<sup>14</sup>See 2.1(g)

**PROPOSITION 3.2.** *The following three conditions are equivalent for a digraph  $G := (X, E)$ :*

- (a)  $G$  is an  $\omega$ -model of  $\text{ZF}_{\text{fin}}$ .
- (b)  $G$  is an  $\omega$ -model of Extensionality, Empty set, Regularity, Adjunction and  $\neg$  Infinity.
- (c)  $G$  satisfies the following four conditions:
  - (i)  $E$  is extensional;
  - (ii) Every vertex of  $G$  has finite in-degree;
  - (iii)  $G$  is acyclic; and
  - (iv)  $G$  has an element of in-degree 0, and for all positive  $n \in \omega$ ,

$$(X, E) \models \forall x_1 \cdots \forall x_n \exists y \forall z \left( zEy \leftrightarrow \bigvee_{i=1}^n z = x_i \right).$$

**PROOF.** (a)  $\Rightarrow$  (b): Trivial.

(b)  $\Rightarrow$  (c): Assuming (b), (i) and (ii) are trivially true. (iv) is an easy consequence of Empty set and  $n$ -applications of Adjunction. To verify (iii), suppose to the contrary that  $x_1 E x_2 \cdots E x_{n-1} E x_n$  is a cycle in  $G$  with  $x_1 = x_n$ . By (iv), there is an element  $y \in X$  with  $y_E = \{x_i : 1 \leq i \leq n\}$ . This contradicts Regularity since such a  $y$  has no minimal “element”.

(c)  $\Rightarrow$  (a): Routine, but we briefly comment on the verification of Regularity, which is accomplished by contradiction: if  $x$  is a nonempty set with no minimal element, then there exists an external infinite sequence  $\langle x_n : n \in \omega \rangle$  of elements of  $c$  such that  $x_{n+1} E x_n$  for all  $n \in \omega$ . Invoking statement (ii) of (c), this shows that there are  $x_m$  and  $x_n$  with  $m < n$  such that  $x_m = x_n$ , which contradicts condition (iii) of (c).  $\dashv$

The following list of definitions prepares the way for the first key theorem of this section (Theorem 3.4), which will enable us to build plenty of  $\omega$ -models of  $\text{ZF}_{\text{fin}}$ .

**DEFINITION 3.3.** Suppose  $G := (X, E)$  is an extensional, acyclic digraph, all of whose vertices have finite in-degree.

(a) A subset  $S$  of  $X$  is said to be *coded* in  $G$  if there is some  $x \in X$  such that  $S = x_E$ .

(b)  $D(G) := \{S \subseteq X : S \text{ is finite and } S \text{ is not coded in } G\}$ . We shall refer to  $D(G)$  as the *deficiency set* of  $G$ .

(c) The infinite sequence of digraphs

$$\langle \mathbb{V}_n(G) : n \in \omega \rangle,$$

where  $\mathbb{V}_n(G) := (V_n(G), E_n(G))$ , is built recursively using the following

clauses<sup>15</sup>:

- $V_0(G) := X; E_0(G) := E;$
- $V_{n+1}(G) := V_n(G) \cup D(\mathbb{V}_n(G));$
- $E_{n+1}(G) := E_n(G) \cup \{\langle x, S \rangle \in V_n(G) \times D(\mathbb{V}_n(G)) : x \in S\}.$

(d)  $\mathbb{V}_\omega(G) := (V_\omega(G), E_\omega(G))$ , where

$$V_\omega(G) := \bigcup_{n \in \omega} V_n(G), \quad E_\omega(G) := \bigcup_{n \in \omega} E_n(G).$$

**THEOREM 3.4.** *If  $G := (X, E)$  is an extensional, acyclic digraph, all of whose vertices have finite in-degree, then  $\mathbb{V}_\omega(G)$  is an  $\omega$ -model of  $\text{ZF}_{\text{fin}}$ .*

**PROOF.** We shall show that  $\mathbb{V}_\omega(G)$  satisfies the four conditions of (c) of Proposition 3.2. Before doing so, let us make an observation that is helpful for the proof, whose verification is left to the reader (footnote 15 comes handy here).

**OBSERVATION 3.4.1.** *For each  $n \in \omega$ ,  $\mathbb{V}_{n+1}(G)$  “end extends”  $\mathbb{V}_n(G)$ , i.e., if  $aE_{n+1}b$  and  $b \in V_n(G)$ , then  $a \in V_n(G)$ .*

It is easy to see that extensionality is preserved in the passage from  $\mathbb{V}_n(G)$  to  $\mathbb{V}_{n+1}(G)$ . By the above observation, at no point in the construction of  $\mathbb{V}_\omega(G)$  a new member is added to an old member, which shows that extensionality is preserved in the limit. It is also easy to see that every vertex of  $\mathbb{V}_\omega(G)$  has finite in-degree. To verify that  $\mathbb{V}_\omega(G)$  is acyclic it suffices to check that each finite approximation  $\mathbb{V}_n(G)$  is acyclic, so we shall verify that it is impossible for  $\mathbb{V}_{n+1}(G)$  to have a cycle and for  $\mathbb{V}_n(G)$  to be acyclic. So suppose there is a cycle  $\langle s_i : 1 \leq i \leq k \rangle$  in  $\mathbb{V}_{n+1}(G)$ , and  $\mathbb{V}_n(G)$  is acyclic. Then by Observation 3.4.1, for some  $i$ ,

$$s_i \in V_{n+1}(G) \setminus V_n(G).$$

This implies that  $s_i$  is a member of the deficiency set of  $\mathbb{V}_n(G)$ , and so  $s_i \subseteq V_n(G)$ . But for  $j = i + 1 \pmod{k}$ , we have  $s_i E_{n+1} s_j$ . This contradicts the definition of  $E_{n+1}$ , thereby completing our verification that  $G$  is acyclic.  $\dashv$

**REMARK 3.5.** (a) The proof of Theorem 3.4 makes it clear that  $G$  is end extended by  $\mathbb{V}_\omega(G)$ ; and every element of  $\mathbb{V}_\omega(G)$  is first order definable in the structure  $(\mathbb{V}_\omega(G), c)_{c \in X}$ .

(b) Recall from part (i) of Definition 2.1 that  $\tau(n, c)$  denotes the  $n$ -th approximation to the transitive closure  $\tau(c)$  of  $\{c\}$ . For any  $c \in V_\omega(G)$ , a tail of  $\tau(n, c)$  lies in  $G$ , i.e.,  $\tau(c) \setminus \tau(n, c) \subseteq G$  for sufficiently large  $n$ .

<sup>15</sup>Since we want the elements of  $X$  to behave like *urelements*, something could go wrong with this definition if some vertex happens to be a finite set of vertices, or a finite set of finite sets of vertices, etc. A simple way to get the desired effect is to replace  $X$  with  $X^* = \{\{\{x\}, X\} : x \in X\}$ . Then  $X^* \cap D(V_n(G)) = \emptyset$  holds for all  $n \in \omega$ , and all digraphs  $G$  with vertex-set  $X^*$ .

EXAMPLE 3.6. (a) For every transitive  $S \subseteq V_\omega$ ,  $\mathbb{V}_\omega(S, \in) \cong (V_\omega, \in)$ .  
 (b) Let  $G_\omega := (\omega, \{\langle n+1, n \rangle : n \in \omega\})$ .  $\mathbb{V}_\omega(G_\omega)$  is our first concrete example of a nonstandard  $\omega$ -model of  $\text{ZF}_{\text{fin}}$ .

COROLLARY 3.7.  $\text{ZF}_{\text{fin}}$  has  $\omega$ -models in every infinite cardinality.

PROOF. For any (finite or infinite) set  $I$ , and any digraph  $G = (X, E)$ , let  $I \times G := (I \times X, F)$ , where

$$\langle i, x \rangle F \langle j, y \rangle \Leftrightarrow i = j \wedge xEy.$$

Note that  $I \times G$  is the disjoint union of  $|I|$  copies of  $G$ . It is easy to see that if  $G$  is an extensional, acyclic digraph, all of whose vertices have finite in-degree, and  $G$  has no vertex  $v$  with  $v_E = \emptyset$ , then  $I \times G$  shares the same features. Therefore if  $G_\omega$  is as in Example 3.6(b), and  $I$  is infinite, then  $\mathbb{V}_\omega(I \times G_\omega)$  is an  $\omega$ -model of  $\text{ZF}_{\text{fin}}$  of the same cardinality as  $I$ .  $\dashv$

REMARK 3.8. Corollary 3.7 shows that in contrast with  $\text{ZF}_{\text{fin}} + \text{TC}$ , within  $\text{ZF}_{\text{fin}}$  there is no definable bijection between the universe and the set of natural numbers. Furthermore, since  $\mathbb{V}_\omega(\{0, 1\} \times G_\omega)$  has an automorphism of order 2, there is not even a definable linear ordering of the universe available in  $\text{ZF}_{\text{fin}}$ .

We need to introduce a key definition before stating the next result:

- A digraph  $G = (\omega, E)$  is said to be *highly recursive* if (1) for each  $n \in \omega$ ,  $n_E$  is finite; (2) The map  $n \mapsto c(n_E)$  is recursive, where  $c$  is a canonical code<sup>16</sup> for  $n_E$ ; and (3)  $\{c(n_E) : n \in \omega\}$  is recursive.

COROLLARY 3.9.  $\text{ZF}_{\text{fin}}$  has nonstandard highly recursive  $\omega$ -models.

PROOF. The constructive nature of the proof of Theorem 3.4 makes it evident that if  $G = (\omega, E)$  is highly recursive digraph, then there is highly recursive  $R \subseteq \omega^2$  such that  $(\omega, R) \cong \mathbb{V}_\omega(G)$ . Since the digraph  $G_\omega$  of Example 3.6(b) is easily seen to be highly recursive, we may conclude that there is a highly recursive  $F \subseteq \omega^2$  such that  $(\omega, F) \cong \mathbb{V}_\omega(G_\omega)$ .  $\dashv$

REMARK 3.10. (a) Generally speaking, if  $\mathfrak{M}$  is a highly recursive  $\omega$ -model of  $\text{ZF}_{\text{fin}}$ , then the set of natural numbers  $\mathbb{N}^{\mathfrak{M}}$  is also recursive. However, it is easy to construct a recursive  $\omega$ -model of  $\text{ZF}_{\text{fin}}$  in which the set of natural numbers  $\mathbb{N}^{\mathfrak{M}}$  is not recursive. With more effort, one can even build a recursive  $\omega$ -model of  $\text{ZF}_{\text{fin}}$  that is not isomorphic to a recursive model with a recursive set of natural numbers.

(b) A minor modification of the proof of Corollary 3.9 shows that there are infinitely many pairwise elementarily inequivalent highly recursive models of  $\text{ZF}_{\text{fin}}$ . This is based on the observation that each digraph in the family  $\{I \times G_\omega : |I| \in \omega\}$  has a highly recursive copy, and for each positive  $n \in \omega$

<sup>16</sup>For example,  $c$  can be defined via  $c(X) = \sum_{n \in X} 2^n$ .

the sentence

$$\exists x_1 \cdots \exists x_n \left( \bigwedge_{1 \leq i \leq n} \text{“}\tau(x_i) \text{ is infinite”} \wedge \bigwedge_{1 \leq i < j \leq n} \tau(x_i) \cap \tau(x_j) = \emptyset \right)$$

holds in  $\mathbb{V}_\omega(I \times G_\omega)$  iff  $|I| \geq n$ . In particular, this shows that in contrast to PA and  $\text{ZF}_{\text{fin}} + \text{TC}$ ,  $\text{ZF}_{\text{fin}} + \neg \text{TC}$  has infinitely many nonisomorphic recursive models. However, as shown by the next theorem,  $\text{ZF}_{\text{fin}}$  does not entirely escape the reach of Tennenbaum phenomena.<sup>17</sup>

**THEOREM 3.11.** *Every recursive model of  $\text{ZF}_{\text{fin}}$  is an  $\omega$ -model.*

**PROOF.** The theory  $\text{ZF}_{\text{fin}}$  is existentially rich (see [Sch-2, Definition 1.1]) as is shown by the recursive sequence  $\langle \theta_n(x) : n \in \omega \rangle$ , where  $\theta_n(x)$  is the existential formula

$$\begin{aligned} \exists x_0, x_1, \dots, x_{n+2}, y_0, y_1, \dots, y_{n+2} \left( \bigwedge_{i < j \leq n+2} x_i \neq x_j \right. \\ \left. \wedge \bigwedge_{i \leq n+2} (x_i \in y_i \wedge x_{i+1} \in y_i \in x) \right). \end{aligned}$$

Here, we understand  $x_{n+3}$  to be  $x_0$ . To see that the  $\theta_n(x)$  are existentially rich, let  $i_0, i_1, i_2, \dots$  be sufficiently fast growing (letting  $i_n = \frac{n^2+5n}{2}$  is good enough) and then let

$$E_n = \{\{i, i+1\} : i_n \leq i < i_n + n - 1\} \cup \{i_n + n - 1, i_n\}.$$

If  $I$  is a finite subset of  $\omega$  and  $a = \bigcup_{n \in I} E_n$ , then

$$\text{ZF}_{\text{fin}} \vdash \theta_n(a) \text{ if } n \in I, \text{ and } \text{ZF}_{\text{fin}} \vdash \neg \theta_n(a) \text{ if } n \notin I.$$

Moreover, thanks to the coding apparatus available in  $\text{ZF}_{\text{fin}}$ , there is a binary formula  $\theta(n, x)$  such that for each  $n \in \omega$ ,

$$\text{ZF}_{\text{fin}} \vdash \forall x (\theta(n, x) \leftrightarrow \theta_n(x)). \quad (*)$$

Suppose  $\mathfrak{M}$  is a non  $\omega$ -model of  $\text{ZF}_{\text{fin}}$ , and let  $A, B$  be two disjoint, recursively inseparable, recursively enumerable sets. Since  $\mathfrak{M}$  satisfies  $\text{Ind}_{\mathbb{N}}$ , we can use Overspill to arrange a definable subset  $S$  of  $\mathfrak{M}$  such that  $A \subseteq S$  and  $B$  is disjoint from  $S$ . If  $S(x)$  is a defining formula for  $S$  in  $\mathfrak{M}$ , then for each  $n \in \omega$ , there is  $c$  in  $\mathfrak{M}$  such that

$$\mathfrak{M} \models \forall i < n (\theta_i(c) \leftrightarrow S(i)).$$

We can use Overspill again and  $(*)$  to obtain a nonstandard  $H \in \mathbb{N}^{\mathfrak{M}}$  and some  $d$  in  $\mathfrak{M}$  such that

$$\mathfrak{M} \models \forall i < H (\theta(i, d) \leftrightarrow i \in d).$$

<sup>17</sup>The proof of Theorem 3.11 indeed shows that every recursive model of EST is an  $\omega$ -model.

This shows that for all  $i \in \omega$ ,  $\mathfrak{M} \models (\theta_i(d) \leftrightarrow i \in d)$ . Therefore if  $\mathfrak{M}$  is recursive, then so is the set of standard numbers in  $d$ , contradicting the recursive inseparability of  $A, B$ .  $\dashv$

Before presenting the next result, let us observe that *the first order theory of a digraph  $G$  does not, in general, determine the first order theory of  $\mathbb{V}_\omega(G)$* . To see this, let  $2 \times G_\omega$  be the disjoint sum of two copies of the digraph  $G_\omega$  of Example 3.6(b). Then  $G_\omega$  and  $2 \times G_\omega$  are elementarily equivalent, but  $\mathbb{V}_\omega(G_\omega)$  and  $\mathbb{V}_\omega(2 \times G_\omega)$  are not elementarily equivalent, indeed, both  $\mathbb{V}_\omega(G_\omega)$  and  $\mathbb{V}_\omega(2 \times G_\omega)$  are the unique  $\omega$ -models of their own first order theories up to isomorphism. However, Theorem 3.12 shows that the  $L_{\infty\omega}$ -theory of  $\mathbb{V}_\omega(G)$  is determined by the  $L_{\infty\omega}$ -theory of  $G$ .

Recall that the infinitary logic  $L_{\infty\omega}$  is the extension of first logic that allows the formation of disjunctions and conjunctions of arbitrary cardinality, but which only uses finite strings of quantifiers. By a classical theorem of Karp [Ba, Ch. VII, Thm. 5.3], two structures  $\mathfrak{A}$  and  $\mathfrak{B}$  are  $L_{\infty\omega}$  equivalent iff  $\mathfrak{A}$  and  $\mathfrak{B}$  are *partially isomorphic*, i.e., there is a collection  $I$  of partial isomorphisms between  $\mathfrak{A}$  and  $\mathfrak{B}$  with the back-and-forth-property (written  $I : \mathfrak{A} \cong_p \mathfrak{B}$ ). More specifically,  $I : \mathfrak{A} \cong_p \mathfrak{B}$  if each  $f \in I$  is an isomorphism between some substructure  $\mathfrak{A}_f$  of  $\mathfrak{A}$  and some substructure  $\mathfrak{B}_f$  of  $\mathfrak{B}$ ; and for every  $f \in I$  and every  $a \in A$  (or  $b \in B$ ) there is a  $g \in I$  with  $f \subseteq g$  and  $a \in \text{dom}(g)$  (or  $b \in \text{ran}(g)$ , respectively).

**THEOREM 3.12.** *Suppose  $G$  and  $G'$  are digraphs with  $G \equiv_{\infty\omega} G'$ . Then  $\mathbb{V}_\omega(G) \equiv_{\infty\omega} \mathbb{V}_\omega(G')$ .*

**PROOF.** By Karp's Theorem, it suffices to show that if  $G \cong_p G'$ , then  $\mathbb{V}_\omega(G) \cong_p \mathbb{V}_\omega(G')$ .<sup>18</sup> Let  $I : G \cong_p G'$  and consider the sequence  $\{I_n : n \in \omega\}$ , with  $I_0 := I$  and  $I_{n+1} = \{\bar{f} : f \in I_n\}$ , where

$$\bar{f} := f \cup \{(S, f(S)) : S \in D(\mathbb{V}_n(G)) \cap \text{dom}(f)\}.$$

We claim that for each  $n \in \omega$ ,  $I_n : \mathbb{V}_n(G) \cong_p \mathbb{V}_n(G')$ . Observe that if  $f \in I_n : \mathbb{V}_n(G) \cong_p \mathbb{V}_n(G)$ , and  $S \in D(\mathbb{V}_n(G))$ , then  $f(S) \in D(\mathbb{V}_n(G'))$ . Using this observation it is easy to show that each member of  $I_n$  is a partial isomorphism between a substructure of  $\mathbb{V}_n(G)$  and a substructure of  $\mathbb{V}_n(G')$ .

The back-and-forth property of  $I_n$  is established by induction. Since the base case is true by our choice of  $I_0$ , and  $G$  and  $G'$  play a symmetric role in the proof, it suffices to verify the “forth” portion of the inductive clause by showing that the forth-property is preserved in the passage from  $I_n$  to  $I_{n+1}$ . So suppose that  $\bar{f} \in I_{n+1}$  (where  $f \in I_n$ ), and  $a \in V_{n+1}(G)$ . We need to find

<sup>18</sup>Advanced methods in set theory provide a succinct proof of this fact. If  $G \cong_p G'$ , then there is a Boolean extension  $V^\mathbb{B}$  of the universe of set theory wherein  $G \cong G'$ , which in turn implies that  $\mathbb{V}_\omega[G] \cong \mathbb{V}_\omega[G']$  in  $V^\mathbb{B}$ . But since  $\mathbb{V}_\omega[G]$  is absolute in the passage to a Boolean extension for every  $G$ , and  $L_{\infty\omega}$ -equivalence is a  $\Pi_1$ -notion,  $\mathbb{V}_\omega[G] \equiv_{\infty\omega} \mathbb{V}_\omega[G']$  in the real world.

$g \in I_n$  such that  $\bar{g}$  extends  $\bar{f}$  and  $a \in \text{dom}(\bar{g})$ . We distinguish two cases:

- Case 1:  $a \in V_n(G)$ .
- Case 2:  $a \in V_{n+1}(G) \setminus V_n(G)$ .

If Case 1 holds, then the desired  $g$  exists by the forth-property of  $I_n$ . On the other hand, if Case 2 holds, then  $a$  is one of the deficiency sets of  $V_n(G)$ , and

$$a = \{c_1, \dots, c_k\} \subseteq V_n(G).$$

Therefore by  $k$ -applications of the “forth” property of  $I_n$ , we can find an extension  $g$  of  $f$  with  $\{c_1, \dots, c_k\} \subseteq \text{dom}(g)$ . This will ensure that  $\bar{g}$  extends  $\bar{f}$  and  $a \in \text{dom}(\bar{g})$ . Having constructed the desired  $\{I_n : n \in \omega\}$ , we let  $I_\omega := \bigcup_{n \in \omega} I_n$ . Clearly

$$I_\omega : \mathbb{V}_\omega(G) \cong_p \mathbb{V}_\omega(G'). \quad \dashv$$

EXAMPLE 3.13. Suppose  $I$  and  $J$  are infinite sets and  $G$  is a digraph. Then  $I \times G \cong_p J \times G$ . To see this, we can choose the corresponding  $I$  to consist of (full) isomorphisms between structures of the form  $X \times G$  and  $Y \times G$ , where  $X$  and  $Y$  are finite subsets of  $I$  and  $J$  (respectively) of the same finite cardinality. In particular, for any digraph  $G$

$$\mathbb{V}_\omega(I \times G) \equiv_{\infty\omega} \mathbb{V}_\omega(J \times G).$$

REMARK 3.14. (a) Let  $\mathcal{G}$  be the category whose *objects* are extensional acyclic digraphs all of whose vertices have finite in-degree, and whose *morphisms* are end embeddings, i.e., embeddings  $f : G \rightarrow G'$  with the property that  $f(G) \subseteq_e G'$ ; and let  $\mathcal{G}_{\text{ZF}_{\text{fin}}}$  be the subcategory of  $\mathcal{G}$  whose objects are  $\omega$ -models of  $\text{ZF}_{\text{fin}}$ . Then there is a functor

$$\Phi : \mathcal{G} \rightarrow \mathcal{G}_{\text{ZF}_{\text{fin}}}.$$

Moreover,  $\Phi$  is a retraction (i.e., if  $G \in \mathcal{G}_{\text{ZF}_{\text{fin}}}$ , then  $\Phi(G) \cong G$ ), and the following diagram commutes (in the diagram  $\eta$  is the inclusion map)

$$\begin{array}{ccc} G & \xrightarrow{f} & G' \\ \downarrow \eta & & \downarrow \eta \\ \mathbb{V}_\omega(G) & \xrightarrow{\Phi(f)} & \mathbb{V}_\omega(G') \end{array}$$

$\Phi$  is defined in an obvious manner:  $\Phi(G) := \mathbb{V}_\omega(G)$ ; and for each end embedding  $f : G \rightarrow G'$ ,  $\Phi(f)$  is recursively constructed by  $\Phi(f)(x) = f(x)$  for all  $x \in G$ , and for  $S \in V_{n+1}(G) \setminus V_n(G)$ ,  $\Phi(f)(S) =$  the unique element  $v$  of  $\mathbb{V}_\omega(G')$  such that  $v_F = \{f(x) : x \in S\}$ , where  $F$  is the membership relation of  $\mathbb{V}_\omega(G')$ . Note that  $\Phi(f)$  is the unique extension of  $f$  to a morphism whose domain is  $\mathbb{V}_\omega(G)$ . Indeed, it is not hard to see that  $\Phi$  is the *left adjoint* of the functor  $e$  that identically embeds  $\mathcal{G}_{\text{ZF}_{\text{fin}}}$  into  $\mathcal{G}$ . Since each functor has at most one left adjoint up to natural isomorphism [Mac, Cor. 1, Ch. IV], this shows

that  $\Phi$  is a closure operation that can be characterized in an “implementation-free” manner.

(b) It is easy to see that for each morphism  $f : G \rightarrow G'$  of  $\mathcal{G}$ ,  $\Phi(f)$  is surjective if  $f$  is surjective. Therefore,  $\Phi(f)$  is an automorphism of  $\mathbb{V}_\omega(G)$  if  $f$  is an automorphism of  $G$ . Indeed, for any fixed  $G \in \mathcal{G}$ , the map  $f \mapsto \Phi(f)$  defines a *group embedding* from  $\text{Aut}(G)$  into  $\text{Aut}(\mathbb{V}_\omega(G))$ . Note that  $\Phi(f)$  is the only automorphism of  $\mathbb{V}_\omega(G)$  that extends  $f$ , since if  $g$  is any automorphism of  $\mathbb{V}_\omega(G)$ , then for each  $S \in \mathbb{V}_\omega(G) \setminus G$ , by Extensionality,  $g(S) = \{g(x) : x \in S\}$ . In general,  $\text{Aut}(G)$  need not be isomorphic to  $\text{Aut}(\mathbb{V}_\omega(G))$ , e.g.,  $G_\omega$  of Example 3.6(b) is a rigid digraph, but  $\text{Aut}(\mathbb{V}_\omega(G_\omega))$  is an infinite cyclic group. However, Theorem 4.2 shows that  $\text{Aut}(G) \cong \text{Aut}(\mathbb{V}_\omega(G))$  for a wide class of digraphs  $G$ .

**§4. Models with special properties.** In this section we refine the method introduced in the proof of Theorem 3.4 in order to construct various large families of  $\omega$ -models of  $\text{ZF}_{\text{fin}}$  with a variety of additional structural features, e.g., having a prescribed automorphism group, or being pointwise definable. The central result of this section is Theorem 4.2 below which shows that one can canonically code any prescribed graph  $(A, F)$ , where  $F$  is a set of 2-element subsets of  $A$ , into a model of  $\text{ZF}_{\text{fin}}$ , thereby yielding a great deal of control over the resulting  $\omega$ -models of  $\text{ZF}_{\text{fin}}$ .

**DEFINITION 4.1.** In what follows, every digraph  $G = (X, E)$  we consider will be such that  $\omega \subseteq X$ ; and for  $n \in \omega$ ,  $n_E = \{0, 1, \dots, n-1\}$ . In particular,  $0_E = \emptyset$ .

(a) Let  $\varphi_1(x)$  be the formula in the language of set theory that expresses “there is a sequence  $\langle x_n : n < \omega \rangle$  with  $x = x_0$  such that  $x_n = \{n, x_{n+1}\}$  for all  $n \in \omega$ ”.

At first sight, this seems to require an existential quantifier over infinite sequences (which is not possible in  $\text{ZF}_{\text{fin}}$ ), but this problem can be circumvented by writing  $\varphi_1(x)$  as the sentence that expresses the equivalent statement

“for every positive  $n \in \omega$ , there is a sequence  $s_n = \langle x_i : i < n \rangle$  with  $x_0 = x$  such that for all  $i < n-1$ ,  $x_i = \{i, x_{i+1}\}$ ”.

(b) Using the above circumlocution, let  $\theta(x, y, e)$  be the formula in the language of set theory that expresses

“ $x \neq y \wedge \varphi_1(x) \wedge \varphi_1(y)$ , with corresponding sequences  $\langle x_n : n \in \omega \rangle$  and  $\langle y_n : n \in \omega \rangle$ , and there is a sequence  $\langle e_n : n \in \omega \rangle$  with  $e = e_0$  such that  $e_n = \{e_{n+1}, x_n, y_n\}$  for all  $n \in \omega$ .”

Then let  $\varphi_2(x, y) := \exists e \theta(x, y, e)$ .

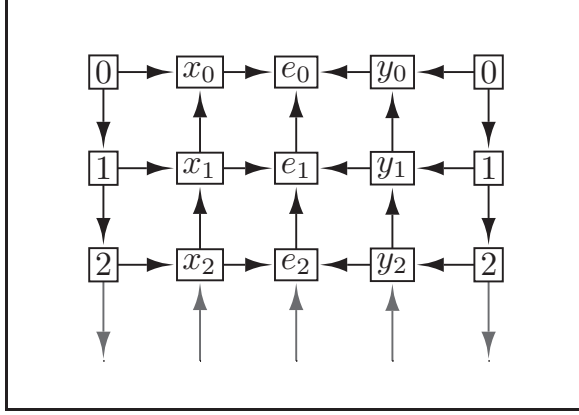


FIGURE 1. Representation of an Edge.

**THEOREM 4.2.** *For every graph  $(A, F)$  there is an  $\omega$ -model  $\mathfrak{M}$  of  $\text{ZF}_{\text{fin}}$  whose universe contains  $A$  and which satisfies the following conditions:*

- (a)  $(A, F)$  is definable in  $\mathfrak{M}$ ;
- (b) Every element of  $\mathfrak{M}$  is definable in  $(\mathfrak{M}, x)_{x \in A}$ ;
- (c) If  $(A, F)$  is pointwise definable, then so is  $\mathfrak{M}$ ;
- (d)  $\text{Aut}(\mathfrak{M}) \cong \text{Aut}(A, F)$ .

**PROOF.** Let

$$X = \omega \cup ((A \cup F) \times \omega).$$

For elements of  $X \setminus \omega$ , we write  $z_n$  instead of  $\langle z, n \rangle$ . This notation should be suggestive of how these elements are used in the definitions of  $\varphi_1(x, y)$  and  $\theta(x, y, e)$ . Let

$$\begin{aligned} E = & \{ \langle m, n \rangle : m < n \in \omega \} \cup \\ & \{ \langle n, x_n \rangle : n \in \omega, x \in A \} \cup \\ & \{ \langle x_{n+1}, x_n \rangle : n \in \omega, x \in A \} \cup \\ & \{ \langle e_{n+1}, e_n \rangle : n \in \omega, e \in F \} \cup \\ & \{ \langle x_n, e_n \rangle : n \in \omega, x \in e \in F \}. \end{aligned}$$

Moreover, in order to arrange  $A \subseteq X$  we can identify elements of the form  $\langle x, 0 \rangle$  with  $x$  when  $x \in A$ . This gives  $G = (X, E)$ , which is an extensional, acyclic digraph all of whose vertices have finite in-degree. Let  $\mathfrak{M} := \mathbb{V}_\omega(G)$ . By Theorem 3.4,  $\mathfrak{M}$  is an  $\omega$ -model of  $\text{ZF}_{\text{fin}}$ . Note that if  $n \in \omega$ ,  $x \in A$ , and  $e = \{x, y\} \in F$ , then  $\mathfrak{M}$  satisfies both “ $x_n = \{x_{n+1}, n\}$ ” and “ $e_n = \{x_n, y_n, e_{n+1}\}$ ”, as shown in Figure 1.

It is clear that if  $\{x, y\} \in F$ , then  $\mathfrak{M} \models \varphi_2(x, y)$ . In order to establish (a) it suffices to check that if  $\mathfrak{M} \models \varphi_1(x)$ , then  $x \in A$ , i.e., we need to verify that  $V_\omega(G) \setminus X$  does not contain an element that satisfies  $\varphi_1$ . Suppose  $\mathfrak{M} \models \varphi_1(c)$  for some  $c \in V_\omega(G) \setminus X$ . Then there is a sequence  $\langle c_n : n < \omega \rangle$  with  $c = c_0$  such that  $c_n = \{n, c_{n+1}\}$  for all  $n \in \omega$ . On the other hand, by Remark 3.5(b), we may choose  $n_0 > 0$  as the first  $n \in \omega$  for which  $\tau(c) \setminus \tau(n, c) \subseteq G$ . It is easy to see that this allows us to find some  $x \in A$  such that  $\tau(c) \setminus \tau(n, c) = \tau(x) \setminus \tau(n, x)$ . Since  $n_0 > 0$  this in turn shows that  $\{n_0, c_{n_0+1}\} = \{n_0, x_{n_0+1}\}$  which implies that  $\tau(c) \setminus \tau(n_0 - 1, c) \subseteq G$ , thereby contradicting the minimality of  $n_0$ . This concludes the proof of (a).

In light of Remark 3.5(a), in order to establish (b) it suffices to show that every vertex of  $G$  is definable in  $(\mathfrak{M}, x)_{x \in A}$ . Since it is clear that each element of  $\omega$  is definable in  $\mathfrak{M}$ , we shall focus on the definability of elements of  $(A \cup F) \times \omega$ . Each  $x_n$  is definable in  $(\mathfrak{M}, x_0)$  since  $\mathfrak{M}$  satisfies “ $x_n = \{n, x_{n+1}\}$ ”. Similarly, since  $\mathfrak{M}$  satisfies “ $e_n = \{e_{n+1}, x_n, y_n\}$ ” the definability of each  $e_n$  follows from the definability of  $e_0$  in  $(\mathfrak{M}, x)_{x \in A}$ . To verify this, suppose  $e = \{x, y\}$ . Let  $\theta$  be as in Definition 4.1. Then  $\theta(x, y, z)$  defines  $e_0$  in  $(\mathfrak{M}, x, y)$ .

(c) is an immediate consequence of (a) and (b) since if  $(A, F)$  is pointwise definable, then by (a), every element of  $A$  is definable in  $\mathfrak{M}$ .

To prove (d), we first establish  $\text{Aut}(G) \cong \text{Aut}(\mathfrak{M})$ , where  $G$  is the digraph  $(X, E)$ . We already commented in Remark 3.14(b) that (1) there is an embedding  $\Phi$  from  $\text{Aut}(G)$  into  $\text{Aut}(\mathfrak{M})$ , and (2) every automorphism of  $G$  has a *unique* extension to an automorphism of  $\mathfrak{M}$ . Therefore, in order to show that  $\Phi$  is surjective, it suffices to show that  $X$  is definable in  $\mathfrak{M}$  since the definability of  $X$  in  $\mathfrak{M}$  would imply that  $g \upharpoonright X \in \text{Aut}(G)$  for every  $g \in \text{Aut}(\mathfrak{M})$ . Recall that  $\varphi_1$  defines  $\{x_0 : x \in A\}$  in  $\mathfrak{M}$ . On the other hand,

$$\varphi_3(z) := \exists x \exists y \theta(x, y, z)$$

defines  $\{e_0 : e \in F\}$  in  $\mathfrak{M}$ . So  $X$  is definable in  $\mathfrak{M}$  by the formula

$$\varphi_4(u) := \mathbb{N}(u) \vee \exists v (u \in \tau(v) \wedge (\varphi_1(v) \vee \varphi_3(v))).$$

This concludes the proof of  $\text{Aut}(G) \cong \text{Aut}(\mathfrak{M})$ . Therefore to establish (c) it suffices to verify that  $\text{Aut}(A, F) \cong \text{Aut}(G)$ . Suppose  $f \in \text{Aut}(A, F)$ . We shall build  $\bar{f} \in \text{Aut}(G)$  such that  $f \mapsto \bar{f}$  describes an isomorphism between  $\text{Aut}(A, F)$  and  $\text{Aut}(G)$ .  $\bar{f}$  is defined by cases:

- $\bar{f}(n) = n$  for  $n \in \omega$ .
- $\bar{f}$  is defined recursively on  $\{x_n : x \in A, n \in \omega\}$  by:  $\bar{f}(x_0) = f(x_0)$  and  $\bar{f}(x_{n+1})$  is the unique element  $v \in G$  for which  $(f(x_n))_E = \{n, v\}$ .
- $\bar{f}$  is defined recursively on  $\{e_n : e \in F, n \in \omega\}$  by: for  $e = \{x, y\}$ ,  $\bar{f}(e_0) = e'_0$ , where  $e' = \{f(x), f(y)\}$ , and  $\bar{f}(e_{n+1})$  is the unique element  $v \in G$  for which  $(\bar{f}(e_n))_E = \{\bar{f}(x_n), \bar{f}(y_n), v\}$ .

Moreover, it is easy to see that if  $g \in \text{Aut}(G)$ , then  $g$  is uniquely determined by  $g \upharpoonright A$ . Therefore in order to verify that the map  $f \mapsto \bar{f}$  is surjective, it suffices to observe that  $A$  is definable in  $G$  by the formula  $\psi(x)$  that expresses “ $0 \in x \wedge |x| = 2 \wedge x \neq \{0, 1\}$ ”. This shows that  $\text{Aut}(A, F) \cong \text{Aut}(G)$  and completes the proof of (d).  $\dashv$

REMARK 4.3. (a) Let  $\mathcal{G}_{\text{ZF}_{\text{fin}}}$  be the category defined in Remark 3.14, and consider the category  $\mathcal{F}$  whose *objects* are graphs  $(A, F)$ , and whose *morphisms* are embeddings  $f : (A, F) \rightarrow (A', F')$ . The proof of Theorem 4.2 shows that there is a functor  $\Psi$  from  $\mathcal{F}$  into  $\mathcal{G}_{\text{ZF}_{\text{fin}}}$  such that for every object  $\mathcal{A} = (A, F)$  of  $\mathcal{F}$ , the set of vertices of  $\Psi(\mathcal{A})$  includes  $A$ ; every automorphism of  $\mathcal{A}$  has a unique extension to an automorphism of  $\Psi(\mathcal{A})$ ; and every automorphism of an object in  $\mathcal{G}_{\text{ZF}_{\text{fin}}}$  of the form  $\Psi(\mathcal{A})$  is uniquely determined by its restriction to  $A$ .

(b) Let  $\varphi_1$  and  $\theta$  be as in Definition 4.1, and let  $\delta(z)$  be the formula in the language of set theory that expresses:

“ $z$  is of the form  $\{X, E\}$ , where  $\varphi_1(x)$  holds for each  $x \in X$ , and for each  $e \in E$ , there are  $x$  and  $y$  in  $X$  such that  $\theta(x, y, e)$ ”,

and let  $\sigma := \forall t \exists z (t \in \tau(z) \wedge \delta(z))$ . Then the class of  $\omega$ -models of  $\text{ZF}_{\text{fin}} + \sigma$  are precisely those objects in  $\mathcal{F}$  that lie in the range of the functor  $\Psi$  defined in (a) above. Note that  $(V_\omega, \in) \models \sigma$  since  $\delta(\{\emptyset\})$  vacuously holds in  $(V_\omega, \in)$ .

COROLLARY 4.4. *Every group can be realized as the automorphism group of an  $\omega$ -model of  $\text{ZF}_{\text{fin}}$ .*<sup>19</sup>

PROOF. A classical theorem of Frucht [Fr] shows that every group can be realized as the automorphism group of a graph.<sup>20</sup>  $\dashv$

COROLLARY 4.5. *For every infinite cardinal  $\kappa$  there are  $2^\kappa$  nonisomorphic rigid  $\omega$ -models of  $\text{ZF}_{\text{fin}}$  of cardinality  $\kappa$ .*

PROOF. It is well-known that there are  $2^\kappa$  nonisomorphic rigid graphs of cardinality  $\kappa$ . One way to see this is to first show that there are  $2^\kappa$  nonisomorphic rigid linear orders of cardinality  $\kappa$  by the following construction: start with the well-ordering  $(\kappa, \in)$ , and for each  $S \subseteq \kappa$  let  $\mathbb{L}_S$  be the linear order obtained by inserting a copy of  $\omega^*$  (the order-type of negative integers) between  $\alpha$  and  $\alpha + 1$  for each  $\alpha \in S$ . Since  $\mathbb{L}_S$  and  $\mathbb{L}_{S'}$  are nonisomorphic for  $S \neq S'$  this completes the argument since every linear order can be coded into a graph with the same automorphism group: given a linear order  $(L, <_L)$ , let  $[L]^2$  be the set of all 2-element subsets of  $L$ , and for each  $s \in [L]^2$ , introduce

<sup>19</sup>In contrast, the class of groups that arise as automorphism groups of models of  $\text{ZF}_{\text{fin}} + \text{TC}$  are precisely the right-orderable groups. This is a consequence of coupling the bi-interpretability of  $\text{ZF}_{\text{fin}} + \text{TC}$  and PA with a key result [KS, Theorem 5.4.4] about automorphisms of models of PA.

<sup>20</sup>See [Lo] for an exposition of Frucht’s theorem and its generalizations.

distinct vertices  $\{a_s, b_s\}$ , and consider the graph  $(A, F)$ , where

$$A = L \cup \{a_s : s \in S\} \cup \{b_s : s \in S\},$$

and

$$\begin{aligned} F = & \{\{x, a_s\} : x \in s \in [L]^2\} \cup \\ & \{\{a_s, b_s\} : s \in [L]^2\} \cup \\ & \{\{x, b_s\} : s = \{x, y\}, \text{ and } x <_L y\}. \end{aligned} \quad \dashv$$

**COROLLARY 4.6.** *For every infinite cardinal  $\kappa$  there is a family  $\mathcal{M}$  of cardinality  $2^\kappa$  of  $\omega$ -models of  $\text{ZF}_{\text{fin}}$  of cardinality  $\kappa$  such that for any distinct  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  in  $\mathcal{M}$ , there is no elementary embedding from  $\mathfrak{M}_1$  into  $\mathfrak{M}_2$ .*

**PROOF.** It is well-known that there is a family  $\mathcal{G}$  of power  $2^\kappa$  of simple graphs of cardinality  $\kappa$  such that for distinct  $G_1$  and  $G_2$  in  $\mathcal{G}$ , there is no embedding from  $G_1$  into  $G_2$ .<sup>21</sup>  $\dashv$

The next corollary is motivated by the following observations: (1)  $(V_\omega, \in)$  is a pointwise definable model, and (2) The Gödel-Rosser incompleteness theorem is powerless in giving any information on the number of complete extensions of  $\text{ZF}_{\text{fin}}$  that possess an  $\omega$ -model.<sup>22</sup>

**COROLLARY 4.7.** *There are  $2^{\aleph_0}$  pointwise definable  $\omega$ -models of  $\text{ZF}_{\text{fin}}$ . Consequently there are  $2^{\aleph_0}$  complete extensions of  $\text{ZF}_{\text{fin}}$  that possess  $\omega$ -models.*

**PROOF.** This is an immediate consequence of Theorem 4.2(c) and the well-known fact that there are  $2^{\aleph_0}$  nonisomorphic pointwise definable graphs. One way to establish the latter fact is as follows: for each  $S \subseteq \omega$  first build a graph  $\mathcal{A}_S$  with the property that no two vertices have the same degree, each vertex has finite degree, and if  $n < \omega$ , then  $n \in S$  iff there is a vertex having degree  $n$ . Clearly  $\mathcal{A}_S$  is pointwise definable, and distinct  $S$ 's yield nonisomorphic  $\mathcal{A}_S$  (incidentally, if  $0 \notin S$ , then  $\mathcal{A}_S$  can be arranged to be connected. Also, if  $S$  is r.e., then a highly recursive  $\mathcal{A}_S$  can be constructed).  $\dashv$

The next corollary is an immediate consequence of coupling Theorem 4.2 with a classical construction [Ho, Theorem 5.5.1] that implies that for every structure  $\mathfrak{A}$  in a *finite* signature there is a graph  $G_{\mathfrak{A}}$  that is bi-interpretable with an isomorphic copy of  $\mathfrak{A}$ . It is easy to see (but a bit cumbersome to write out the details of the proof) that if two structures  $\mathfrak{A}$  and  $\mathfrak{B}$  are bi-interpretable, then (1) their automorphism groups  $\text{Aut}(\mathfrak{A})$  and  $\text{Aut}(\mathfrak{B})$  are

<sup>21</sup>Indeed, by a theorem of Nešetřil and Pultr [NP] one can strengthen this result by replacing “embedding” to “homomorphism”. We should also point out that by a result of Perminov [P] for each infinite cardinal  $\kappa$ , there are  $2^\kappa$  nonisomorphic rigid graphs none of which can be embedded in to any of the others. This shows that Corollaries 4.6 and 4.7 can be dovetailed.

<sup>22</sup>The Gödel-Rosser theorem can be fine-tuned to show the essential undecidability of consistent first order theories that interpret Robinson's Q [HP, Thm. 2.10, Chap. III], thereby showing that such theories have continuum-many consistent completions (indeed, the result continues to hold with Q replaced by the weaker system R).

isomorphic [Ho, Exercise 8, Sec. 5.4], and (2)  $\mathfrak{A}$  is pointwise definable iff  $\mathfrak{B}$  is pointwise definable. We could have used this approach to provide succinct indirect proofs of the above corollaries, but in the interest of perspicuity, we opted for more direct proofs.

**COROLLARY 4.8.** *For every structure  $\mathfrak{A}$  in a finite signature there is an  $\omega$ -model  $\mathfrak{M}$  of  $\text{ZF}_{\text{fin}}$  such that  $\mathfrak{M}$  interprets an isomorphic copy of  $\mathfrak{A}$ , and  $\text{Aut}(\mathfrak{A}) \cong \text{Aut}(\mathfrak{B})$ . Furthermore, if  $\mathfrak{A}$  is pointwise definable, then so is  $\mathfrak{M}$ .*

To motivate the last result of this section, let us recall that  $V_\omega$  is the only  $\omega$ -model of  $\text{ZF}_{\text{fin}} + \text{TC}$  up to isomorphism. As shown by Corollary 4.8 below, there are many other “categorical” *finite* extensions of  $\text{ZF}_{\text{fin}}$ . One can also show that there are continuum-many completions of  $\text{ZF}_{\text{fin}}$  that possess a unique  $\omega$ -model up to isomorphism.

**COROLLARY 4.9.** *There are infinitely many countable nonisomorphic  $\omega$ -models of  $\text{ZF}_{\text{fin}}$  each of which is the unique  $\omega$ -model of some finite extension of  $\text{ZF}_{\text{fin}}$ .*

**PROOF.** For  $0 < r \in \omega$ , let  $\psi_r(x)$  be the following variant of  $\varphi_1(x)$  of Definition 4.1:

“there is a sequence  $\langle x_n : n < \omega \rangle$  with  $x = x_0$  such that  $x_n = \{n, x_{n+1}\}$  if  $r \mid n$ ; and  $x_n = \{x_{n+1}\}$  otherwise.”

Next, let  $\theta_r$  be the sentence that expresses

“ $\exists x (V = V_\omega(\tau(x), \in))$  and  $\psi_r(x)$ ”.

Clearly  $\text{ZF}_{\text{fin}} + \theta_r$  has a unique  $\omega$ -model up to isomorphism, and for  $r \neq s$ , no  $\omega$ -model of  $\text{ZF}_{\text{fin}}$  satisfies both  $\theta_r$  and  $\theta_s$ .  $\dashv$

**REMARK 4.10.** The theories  $\text{ZF}_{\text{fin}} + \theta_r$  of the proof of Corollary 4.9 provide examples of extensions of  $\text{ZF}_{\text{fin}} + \neg\text{TC}$  that are bi-interpretable with PA. Furthermore, for every positive  $r$ , every model of PA has a unique extension to a model of  $\text{ZF}_{\text{fin}} + \theta_r$ , i.e., if  $\mathfrak{M}$  and  $\mathfrak{N}$  are models of  $\text{ZF}_{\text{fin}} + \theta_r$  such that there is an isomorphism  $f$  between  $(\mathbb{N}, +, \times)^{\mathfrak{M}}$  and  $(\mathbb{N}, +, \times)^{\mathfrak{N}}$ , then  $f$  has a (unique) extension to an isomorphism between  $\mathfrak{M}$  and  $\mathfrak{N}$ .

**§5.  $\text{ZF}_{\text{fin}}$  and PA are not bi-interpretable.** In this section we wish to carry out the promise made at the end of Remark 2.2(f) by establishing the strong failure of bi-interpretability of  $\text{ZF}_{\text{fin}}$  and PA. Recall that two theories  $U$  and  $V$  are said to be *bi-interpretable* if there are interpretations  $\mathcal{I} : U \rightarrow V$  and  $\mathcal{J} : V \rightarrow U$ , a binary  $U$ -formula  $F$ , and a binary  $V$ -formula  $G$ , such that  $F$  is,  $U$ -verifiably, an isomorphism between  $\text{id}_U$  and  $\mathcal{J} \circ \mathcal{I}$ , and  $G$  is,  $V$ -verifiably, an isomorphism between  $\text{id}_V$  and  $\mathcal{I} \circ \mathcal{J}$ . This notion is entirely syntactic, but has several model theoretic ramifications. In particular, given models  $\mathfrak{A} \models U$  and  $\mathfrak{B} \models V$ , the interpretations  $\mathcal{I}$  and  $\mathcal{J}$  give rise to (1) models  $\mathcal{J}^{\mathfrak{A}} \models V$  and  $\mathcal{I}^{\mathfrak{B}} \models U$ , and (2) isomorphisms  $F^{\mathfrak{A}}$  and  $G^{\mathfrak{B}}$  with

$$F^{\mathfrak{A}} : \mathfrak{A} \longrightarrow (\mathcal{J} \circ \mathcal{I})^{\mathfrak{A}} = \mathcal{I}(\mathcal{J}^{\mathfrak{A}}) \text{ and } G^{\mathfrak{B}} : \mathfrak{B} \xrightarrow{\cong} (\mathcal{I} \circ \mathcal{J})^{\mathfrak{B}} = \mathcal{J}(\mathcal{I}^{\mathfrak{B}}).$$

A much weaker notion, dubbed *sentential equivalence*<sup>23</sup> in [Vi-2], is obtained by replacing the demand on the existence of definable isomorphisms with the requirement that the relevant models be elementarily equivalent, i.e., for any  $\mathfrak{A} \models U$  and  $\mathfrak{B} \models V$ ,

$$\mathfrak{A} \equiv (\mathcal{J} \circ \mathcal{I})^{\mathfrak{A}} \text{ and } \mathfrak{B} \equiv (\mathcal{I} \circ \mathcal{J})^{\mathfrak{B}}.$$

**THEOREM 5.1.**  $\text{ZF}_{\text{fin}}$  and PA are not sententially equivalent.

**PROOF.** Suppose to the contrary that the interpretations

$$\mathcal{I} : \text{ZF}_{\text{fin}} \rightarrow \text{PA}, \text{ and } \mathcal{J} : \text{PA} \rightarrow \text{ZF}_{\text{fin}}$$

witness the sentential equivalence of  $\text{ZF}_{\text{fin}}$  and PA. In light of the fact that there are at least two elementarily inequivalent recursive models of  $\text{ZF}_{\text{fin}}$ , in order to reach a contradiction it is sufficient to demonstrate that the hypothesis about  $\mathcal{I}$  and  $\mathcal{J}$  can be used to show that any two arithmetical  $\omega$ -models of  $\text{ZF}_{\text{fin}}$  are elementarily equivalent. To this end, suppose  $\mathfrak{M}$  is an arithmetical  $\omega$ -model of  $\text{ZF}_{\text{fin}}$ . We claim that

$$\mathcal{J}^{\mathfrak{M}} \cong (\omega, +, \times).$$

The following classical theorem of Scott<sup>24</sup> plays a key role in the verification of our claim. In what follows, an *arithmetical model* of PA refers to a structure of the form  $(\omega, \oplus, \otimes)$ , where there are first order formulas  $\varphi(x, y, z)$  and  $\psi(x, y, z)$  that respectively define the graphs of the binary operations  $\oplus$  and  $\otimes$  in the model  $(\omega, +, \times)$ .

**THEOREM (Scott [Sco]).** *No arithmetical nonstandard model of PA is elementarily equivalent to  $(\omega, +, \times)$ .*

Let  $\mathfrak{M}' \equiv (\mathcal{J} \circ \mathcal{I})^{\mathfrak{M}}$ . By assumption,  $\mathfrak{M}' \equiv \mathfrak{M}$ , which implies that

$$(\mathbb{N}, +, \times)^{\mathfrak{M}} \equiv (\mathbb{N}, +, \times)^{\mathfrak{M}'}$$

This shows that  $(\mathbb{N}, +, \times)^{\mathfrak{M}'}$  is elementarily equivalent to  $(\omega, +, \times)$  since  $\mathfrak{M}$  is an  $\omega$ -model. Coupled with the fact that  $(\mathbb{N}, +, \times)^{\mathfrak{M}'}$  is an arithmetical model (since it is arithmetically interpretable in an arithmetical model), Scott's aforementioned theorem can be invoked to show that  $\mathfrak{M}'$  must be an  $\omega$ -model. But since no nonstandard model of PA can interpret an isomorphic copy of  $(\omega, <)$  this shows that for any arithmetical  $\omega$ -model  $\mathfrak{M}$  of  $\text{ZF}_{\text{fin}}$ ,  $\mathcal{J}^{\mathfrak{M}} \cong (\omega, +, \times)$ . Therefore, the assumption of sentential equivalence of  $\text{ZF}_{\text{fin}}$  and PA implies that any two arithmetical  $\omega$ -models of  $\text{ZF}_{\text{fin}}$  are elementarily equivalent, which is the contradiction we were aiming to arrive at.  $\dashv$

<sup>23</sup>It is easy to see (using the completeness theorem for first order logic) that sentential equivalence can also be *syntactically* formulated.

<sup>24</sup>In order to make the proof self-contained, we sketch an outline of the proof of Scott's theorem. If a nonstandard model  $\mathfrak{A}$  is elementarily equivalent to  $(\omega, +, \cdot)$ , then the *standard system*  $\text{SSy}(\mathfrak{A})$  of  $\mathfrak{A}$  has to include all arithmetical sets. If, in addition,  $\mathfrak{A}$  were to be arithmetical, then every member of  $\text{SSy}(\mathfrak{A})$ , and therefore every arithmetical set, would have to be of bounded quantifier complexity, contradiction. Scott's result has recently been revisited in [IT, Sec. 3].

REMARK 5.2. (a) The proof of Theorem 5.1 only invoked one of the two elementary equivalences stipulated by the definition of sentential equivalence, namely that  $\mathfrak{M} \equiv (\mathcal{J} \circ \mathcal{I})^{\mathfrak{M}}$  for every  $\mathfrak{M} \models \text{ZF}_{\text{fin}}$ . In the terminology of [Vi-2], this shows that  $\text{ZF}_{\text{fin}}$  is not a retract of PA in the appropriate category in which definitional equivalence is isomorphism.

(b) As mentioned in Remark 3.10(b), there are infinitely many elementarily inequivalent recursive  $\omega$ -models of  $\text{ZF}_{\text{fin}} + \neg \text{TC}$ . Using the proof of Theorem 5.1 this fact can be used to show that the theories  $\text{ZF}_{\text{fin}} + \neg \text{TC}$  and PA are not sententially equivalent either. However, as pointed out in Remark 4.10, there are finite extensions of  $\text{ZF}_{\text{fin}} + \neg \text{TC}$  that are bi-interpretable with PA.

**§6. Concluding remarks and open questions.** Let  $\mathfrak{A} \equiv_n^{\text{Levy}} \mathfrak{B}$  abbreviate the assertion that  $\mathfrak{A}$  and  $\mathfrak{B}$  satisfy the same  $\Sigma_n^{\text{Levy}}$  formulas in the usual Levy hierarchy of formulas of set theory (in which only unbounded quantification is significant).

THEOREM 6.1. *If  $\mathfrak{A}$  and  $\mathfrak{B}$  are  $\omega$ -models of  $\text{ZF}_{\text{fin}}$ , then  $\mathfrak{A} \equiv_1^{\text{Levy}} \mathfrak{B}$ .*

PROOF. We use a variant of the Ehrenfeucht-Fraïssé game adapted to this context, in which once the first player (spoiler) and second player (duplicator) have made their first moves by choosing elements from each of the two structures, the players are obliged to select only members of elements that have been chosen already (by either party). We wish to show that for any particular length  $n$  of the play ( $n > 0$ ), the duplicator has a winning strategy. Assume, without loss of generality, that the spoiler chooses  $a_1 \in A$ . The duplicator responds by choosing  $b_1 \in B$  with the key property that there is function  $f$  such that

$$f : (\tau(n, a), \in)^{\mathfrak{A}} \xrightarrow{\cong} (\tau(n, b), \in)^{\mathfrak{B}}.$$

(it is easy to see, using the fact that  $\mathfrak{A}$  and  $\mathfrak{B}$  are  $\omega$ -models of  $\text{ZF}_{\text{fin}}$ , that the duplicator can always do this). From that point on, once the spoiler picks any element  $c$  from either structure the duplicator responds with  $f(c)$  or  $f^{-1}(c)$  depending on whether  $c$  is in the domain or co-domain of  $f$ .  $\dashv$

Note that TC is  $\Pi_2^{\text{Levy}}$  (since  $\text{TC}(x)$  is  $\Sigma_1^{\text{Levy}}$ ) relative to  $\text{ZF}_{\text{fin}}$ , so in the conclusion of Theorem 6.1,  $\equiv_1^{\text{Levy}}$  cannot be replaced by  $\equiv_2^{\text{Levy}}$  since  $\mathfrak{A}$  can be chosen to be  $(V_\omega, \in)$  and  $\mathfrak{B}$  can be chosen to be a nonstandard  $\omega$ -model of  $\text{ZF}_{\text{fin}}$ . These considerations motivate the next question.

QUESTION 6.2. *Is it true that for each  $n \geq 1$  there are  $\omega$ -models  $\mathfrak{A}$  and  $\mathfrak{B}$  of  $\text{ZF}_{\text{fin}}$  that are  $\equiv_n^{\text{Levy}}$  equivalent but not  $\equiv_{n+1}^{\text{Levy}}$  equivalent?*

QUESTION 6.3. *For infinite cardinals  $\kappa$  and  $\lambda$ , let  $\mathcal{M}(\kappa, \lambda)$  be the class of models  $\mathfrak{M}$  of  $\text{ZF}_{\text{fin}}$  such that the cardinality of  $\mathfrak{M}$  is  $\kappa$ , and the cardinality of  $\mathbb{N}^{\mathfrak{M}}$  is  $\lambda$ .*

(a) Is there a first order scheme  $\Gamma_1$  in the language of set theory such that  $\text{Th}(\mathcal{M}(\omega_1, \omega)) = \text{ZF}_{\text{fin}} + \Gamma_1$ ?

(b) Is there a first order scheme  $\Gamma_2$  in the language of set theory such that  $\text{Th}(\mathcal{M}(\beth_\omega, \omega)) = \text{ZF}_{\text{fin}} + \Gamma_2$ ?

Question 6.3 is motivated by two classical two-cardinal theorems of Model Theory<sup>25</sup> due to Vaught, which show that the answers to both parts of Question 6.3 are positive if “first order scheme” is weakened to “recursively enumerable set of first order sentences”. The aforementioned two-cardinal theorems also show that (1) for  $\kappa > \lambda \geq \omega$ ,  $\text{Th}(\mathcal{M}(\kappa, \lambda)) \supseteq \text{Th}(\mathcal{M}(\omega_1, \omega))$ , and (2) for all  $\kappa \geq \beth_\omega$ ,  $\text{Th}(\mathcal{M}(\kappa, \omega)) \subseteq \text{Th}(\mathcal{M}(\beth_\omega, \omega))$ .

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<sup>25</sup>See [CK, Theorems 3.2.12 and 7.2.2], and [Sch-1].

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# TENNENBAUM'S THEOREM FOR MODELS OF ARITHMETIC

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**Abstract.** This paper discusses Tennenbaum's Theorem in its original context of models of arithmetic, which states that there are no recursive nonstandard models of Peano Arithmetic. We focus on three separate areas: the historical background to the theorem; an understanding of the theorem and its relationship with the Gödel–Rosser Theorem; and extensions of Tennenbaum's theorem to diophantine problems in models of arithmetic, especially problems concerning which diophantine equations have roots in some model of a given theory of arithmetic.

**§1. Some historical background.** The theorem known as “Tennenbaum's Theorem” was given by Stanley Tennenbaum in a paper at the April meeting in Monterey, California, 1959, and published as a one-page abstract in the Notices of the American Mathematical Society [28]. It is easily stated as saying that there is no nonstandard recursive model of Peano Arithmetic, and is an attractive and rightly often-quoted result.

This paper celebrates Tennenbaum's Theorem; we state the result fully and give a proof of it and other related results later. This introduction is in the main historical. The goals of the latter parts of this paper are: to set out the connections between Tennenbaum's Theorem for models of arithmetic and the Gödel–Rosser Theorem and recursively inseparable sets; and to investigate stronger versions of Tennenbaum's Theorem and their relationship to some diophantine problems in systems of arithmetic.

Tennenbaum's theorem was discovered in a period of foundational studies, associated particularly with Mostowski, where it still seemed conceivable that useful independence results for arithmetic could be achieved by a “hands-on” approach to building nonstandard models of arithmetic. Mostowski's own aspirations for the programme are clearly set out in his address to the 8th Congress of Polish mathematicians in September 1953 [16]. Mostowski called for an axiomatic treatment of arithmetic and its models, and cites Ryll–Nardzewski [22] and Rosser and Wang [21] as early proponents of the theory of models of arithmetic. Mostowski places problems of induction and inductive definitions at the forefront of studies in arithmetic, but lists several

“secondary but nevertheless important and interesting problems connected with the axioms of arithmetic of natural numbers”:

What kind of structure have the models of Peano's arithmetic differing from a model composed of natural numbers; in particular, what is their ordinal [i.e. order] type like? After Rosser and Wang [21] we term such models *non-standard*. Some initial results on these lines have been published by Kemeny [10].

To find out whether by using non-standard models it would be possible to obtain proofs for the independence of classical number-theoretical problems of the system of arithmetical axioms.

To prove the incompleteness of axiomatic arithmetic without applying the method of arithmetization by giving suitable models showing the consistency and independence of an appropriately chosen number-theoretical axiom.

Mostowski's questions are good ones. Although the order-type of a countable model of PA is easy to determine, it seems that even to this day almost nothing is known about order-types of uncountable models. Nor have any of the classical number-theoretical problems yielded to logical methods of proving independence. Only the last problem, independence results without arithmetization, has yielded results.

Skolem had, of course, already built nonstandard models of arithmetic by an ultrapower-like construction. His 1955 paper [27] is the most accessible and widely quoted, and was written for the proceedings of a symposium in Amsterdam in 1954, but is essentially just a translation of an earlier (1934) paper [26]. Skolem had raised the question of recursiveness of nonstandard models, and Mostowski answered his question, using recursively inseparable r.e. sets to show that no nonstandard model of primitive recursive arithmetic with predicates for all primitive recursive functions can be recursive [17]. Kreisel's review [*Mathematical Reviews*, MR0093483] of this paper is illuminating:

The proof [of Mostowski's main theorem] shows that there is no recursive non-standard model in which all theorems of (quantifier-free) primitive recursive arithmetic are valid. In other words, recursive models are useless for independence problems in full primitive recursive arithmetic. This answers a question raised by Skolem at Amsterdam in 1954 [...]. It is not known if elementary arithmetic with addition and multiplication as the sole non-logical constants has a recursive non-standard model.

Mostowski's theorem was not an isolated event, but followed from a number of earlier papers constructing theories with no recursive models [14, 15], and Kreisel [12] and Putnam [20] had independently obtained similar results to Mostowski's.

As for independence results, Kemeny did produce a model that addressed some questions of independence in 1958 [11]. As one in hindsight would expect, Kemeny's model was shown soon afterwards to fail to satisfy any particularly interesting induction axioms for arithmetic (Gandy [5]). To my mind, the highlight of this period of building recursive models for the purposes of independence results was the results of the early 1960s by Shepherdson, who, using algebraic methods, produced beautiful nonstandard models of quantifier-free arithmetic in which he showed number theoretic results such as the infinitude of primes and Fermat's Last Theorem (in fact, for exponent 3) are false [25]. By this time Tennenbaum's theorem was already well known, and Shepherdson and others were certainly aware of the limitations of this approach.

Further results along the same lines as Tennenbaum's appeared soon afterwards. Feferman [3] presented a detailed examination of arithmetization, including the result quoted by Tennenbaum in his abstract that no nonstandard model of full arithmetic can be arithmetically defined. Dana Scott [23] also investigated constructions of models of arithmetic, including an elegant algebraic construction and a proof of Tennenbaum's theorem.

In the following sections, we shall go into more detail, stating and proving Tennenbaum's theorem, and then discussing what theory of arithmetic is actually needed in the model for it to hold. This will bring us on to more recent questions connected with diophantine problems and the solution to Hilbert's 10th problem.

**§2. Tennenbaum's theorem.** I will give a presentation of Tennenbaum's theorem and some variations on it here. All models considered here will be countable models of at least  $PA^-$ , the theory of the nonnegative part of discretely ordered rings. Except where explicitly stated otherwise, we will regard the standard model  $\mathbb{N}$  as an initial segment of each nonstandard model of arithmetic  $M$ . We shall assume some basic theory from models of arithmetic, such as coding techniques, etc., and to allow us to switch views from recursion theory on  $\mathbb{N}$  and recursive sets of formulas we identify a formula with its Gödel-number via some natural Gödel-numbering. All notation not explained here is as in *Models of Peano Arithmetic* [7].

A structure  $M = (M, +, \cdot, 0, 1, <)$  is *recursive* if there is a 1–1 correspondence  $f : \mathbb{N} \rightarrow M$  such that, identifying  $\mathbb{N}$  with the domain of  $M$  via  $f$ , the functions  $+$ ,  $\cdot$  and relation  $<$  on  $M$  are recursive on  $\mathbb{N}$ . Note that these functions and relation do not need to correspond to the usual ones on  $\mathbb{N}$ . The structure  $M$  is *non-recursive* if there is no such 1–1 correspondence. The standard model  $(\mathbb{N}, +, \cdot, 0, 1, <)$  with the usual addition, multiplication and order is recursive, via the identity map  $\mathbb{N} \rightarrow \mathbb{N}$ . Tennenbaum showed that this is the only such recursive model of arithmetic.

**THEOREM 2.1** (Tennenbaum, 1959). *Let  $M = (M, +, \cdot, 0, 1, <)$  be a countable model of PA, and not isomorphic to the standard model  $(\mathbb{N}, +, \cdot, 0, 1, <)$ . Then  $M$  is not recursive.*

It turns out that the choice of theory here is rather inessential. Indeed Tennenbaum doesn't bother to state which theory is taken here, simply writing in his abstract "provable", which presumably meant "provable in PA". The theory PA may be replaced by much weaker sub-theories, including some finitely axiomatized sub-theories. How far one can go in this direction is an interesting question that will be discussed later.

Tennenbaum's theorem improved on Mostowski's attempts at proving similar results. The key technique was a suitable choice of coding mechanism in arithmetic.

I will present a proof of Tennenbaum's theorem shortly. Before I do so, I would like to indicate at least one aspect of what it says: in some precise sense Tennenbaum's theorem is a model-theoretic version of the Gödel–Rosser incompleteness theorem.

**DEFINITION 2.2.** For a theory  $T$  in the language of arithmetic, denote by  $\Pi_1 - \text{Th}(T)$  the set of  $\Pi_1$  consequences of  $T$ , i.e. the set of sentences  $\{\sigma \in \Pi_1 : T \vdash \sigma\}$ . Similarly  $\Pi_n - \text{Th}(T)$ ,  $\Sigma_n - \text{Th}(T)$ , etc.

**THEOREM 2.3** (Rosser). *There is no consistent extension  $T$  of PA for which  $\Pi_1 - \text{Th}(T)$  is recursive.*

**PROOF.** I shall sketch a proof assuming Tennenbaum's theorem as stated earlier but for the set of  $\Pi_2$ -consequences of PA, rather than PA itself.

First, assume that  $T \supseteq \text{PA}$  is consistent and  $\Pi_1 - \text{Th}(T)$  is recursive. We make a model in the usual Henkin-style using model-theoretic forcing with  $\Delta_0$  conditions. That is, at stage  $k$  of the construction we have a  $\Delta_0$  condition, i.e. a  $\Delta_0$  formula  $\lambda_k(w_0, \dots, w_{n_k})$  in special "witnessing" constants  $w_i$  which is a conjunction of several  $\Delta_0$  formulas that we want to make true (including all previous conditions in the construction) such that  $T + \lambda_k(w_0, \dots, w_{n_k})$  is consistent. The resulting model  $M$  will be formed from the  $w_i$ , and by usual techniques we can ensure that  $M$  is a  $\Sigma_1$ -elementary submodel of a model  $N$  of  $T$  together with all the  $\lambda_k(w_0, \dots, w_{n_k})$ , so in particular  $M \models \Pi_2 - \text{Th}(T)$ .

The assumption that  $\Pi_1 - \text{Th}(T)$  is recursive allows us to ensure that the whole construction can be carried out recursively. This is because during the construction, we need only decide questions such as, given  $\lambda_k(w_0, \dots, w_{n_k})$  and a new  $\Delta_0$  formula  $\theta(\bar{w}, \bar{x})$ , is  $T + \lambda_k(w_0, \dots, w_{n_k}) + \theta(\bar{w}, \bar{x})$  consistent? This amounts to asking if

$$T \vdash \forall w_0, \dots, w_{n_k}, \bar{x} (\lambda_k(w_0, \dots, w_{n_k}) \rightarrow \neg \theta(\bar{w}, \bar{x})),$$

a question that can be effectively decided by looking at  $\Pi_1 - \text{Th}(T)$ . Thus the construction is recursive, and indeed the sequence of conditions produced by

the construction is also a recursive sequence of  $\Delta_0$  formulas in the witnessing constants.

This means that the resulting model  $M$  is recursive, since it is built from an enumerated set of witnesses  $w_i$  modulo the recursive equivalence relation  $w_i \sim w_j$  when  $w_i = w_j$  is a conjunct of some condition in the construction. The truth of any  $\Sigma_1$  sentence  $\exists x \theta(x)$  can also be determined: on the one hand by seeing if  $\theta(w_i)$  is a conjunct of a condition for some  $w_i$ ; and on the other hand by seeing if some other condition  $\lambda_k(w_0, \dots, w_{n_k})$  together with  $T$  implies  $\forall x \neg \theta(x)$ . Thus the model  $M$  is nonstandard because the truth of  $\Sigma_1$  sentences in the standard model  $\mathbb{N}$  is well-known not to be decidable.

We conclude that if  $T \supseteq \text{PA}$  is consistent and  $\Pi_1 - \text{Th}(T)$  is recursive there is a recursive nonstandard model of  $\Pi_2 - \text{Th}(T)$ , and as  $T$  extends PA this contradicts Tennenbaum's theorem for  $\Pi_2 - \text{Th}(\text{PA})$ .  $\dashv$

The Gödel–Rosser Theorem is well-known to be related to the following classical result of recursion theory, which we will use to prove Theorem 2.1.

**THEOREM 2.4.** *There exist r.e. sets  $A, B \subseteq \mathbb{N}$  which are recursively inseparable, i.e. there is no recursive set  $C \subseteq \mathbb{N}$  such that  $A \subseteq C$  and  $B \cap C = \emptyset$ .*

To prove Theorem 2.1, we will follow Tennenbaum and separate the problem into two subproblems: firstly of saying something about which sets  $A \subseteq \mathbb{N}$  are coded in a model  $M$ , and secondly on the consequences of having nonrecursive sets coded.

**DEFINITION 2.5.** Let  $M$  be a nonstandard model of arithmetic. We define  $\text{SSy}(M)$ , the *standard system of sets coded in  $M$* , to be the set of all  $A \subseteq \mathbb{N}$  such that

$$A = \{n \in \mathbb{N} : M \models \eta(n, \bar{a})\}$$

for some formula  $\eta$  and some  $\bar{a} \in M$ .

In most cases, we may fix a particular formula  $\eta$  appropriately and all sets in the standard system appear for this  $\eta$  and a suitable choice of parameters  $\bar{a}$ . By the induction axioms, this will work in PA for any  $\eta$  for which the following statement is provable for all pairs of finite disjoint sets  $A, B \subseteq \mathbb{N}$ :

$$\exists \bar{x} \left( \bigwedge_{i \in A} \eta(i, \bar{x}) \wedge \bigwedge_{j \in B} \neg \eta(j, \bar{x}) \right)$$

This happens in the particular case when  $\eta(n, x)$  is a first-order formula in the language of PA equivalent to  $\exists y (p_n \cdot y = x)$ , where  $p_0 = 2$ ,  $p_1 = 3$ ,  $p_2 = 5$ , and so on, enumerating the standard primes. Thus, for nonstandard models  $M$  of PA, we have

$$\text{SSy}(M) = \{A \subseteq \mathbb{N} : \exists a \in M \ A = \{n \in \mathbb{N} : M \models \eta(n, a)\}\}.$$

This formulation of  $\text{SSy}(M)$  is particularly useful when studying the complexity of addition in a model.

The following lemma is a straightforward application of induction.

LEMMA 2.6 (Robinson's overspill lemma). *Let  $M$  be a nonstandard model of Peano arithmetic, and suppose  $\bar{a} \in M$  and  $\theta(x, \bar{y})$  is a formula such that  $M \models \theta(n, \bar{a})$  for each  $n \in \mathbb{N}$ . Then there is a nonstandard  $x \in M$  such that  $M \models \theta(x, \bar{a})$ .*

The traditional approach to Tennenbaum's theorem now splits into two parts.

THEOREM 2.7. *Let  $M$  be a nonstandard model of Peano arithmetic. Then  $\text{SSy}(M)$  contains a nonrecursive set.*

PROOF. Let  $A, B \subseteq \mathbb{N}$  be r.e. recursively inseparable sets, given by Theorem 2.4. Then these sets are defined (in  $\mathbb{N}$ ) by  $\Sigma_1$  formulas  $\exists y \alpha(x, y)$  and  $\exists z \beta(x, z)$ , respectively, where  $\alpha$  and  $\beta$  are  $\Delta_0$ . We regard the standard model  $\mathbb{N}$  as an initial segment of  $M$  and note that  $\Sigma_1$  formulas are preserved upwards from initial segments to the larger model. So, by this and the disjointness of  $A, B$  we have, for each  $k \in \mathbb{N}$ ,

$$M \models \forall x, y, z < k \neg(\alpha(x, y) \wedge \beta(x, z)).$$

So by the overspill lemma there is some nonstandard  $\zeta \in M$  with

$$M \models \forall x, y, z < \zeta \neg(\alpha(x, y) \wedge \beta(x, z)).$$

Now let  $C \subseteq \mathbb{N}$  be the set  $C = \{n \in \mathbb{N} : M \models \exists y < \zeta \alpha(n, y)\}$ . By preservation of  $\Sigma_1$  formulas and the nonstandardness of  $\zeta$ , we see immediately that  $C \supseteq A$ , and the above property of  $\zeta$  also shows that  $C \cap B = \emptyset$ . So by our choice of  $A$  and  $B$ ,  $C$  is nonrecursive, as required.  $\dashv$

THEOREM 2.8. *Let  $M$  be a model of Peano arithmetic for which  $\text{SSy}(M)$  contains a nonrecursive set. Then  $(M, +)$  is not recursive.*

PROOF. Let  $C \in \text{SSy}(M)$  be nonrecursive. Then by remarks made earlier there is  $a \in M$  such that

$$C = \{n \in \mathbb{N} : M \models \eta(n, a)\}$$

for the formula  $\eta(n, x)$  which is  $\exists y (p_n y = x)$ . Then if  $+$  in  $M$  were recursive so would  $C$  be, since on input  $n \in \mathbb{N}$  we may compute  $p_n$  (which is the  $n$ th prime in  $M$  just as it is in  $\mathbb{N}$  by preservation between  $M$  and  $\mathbb{N}$ ) and search for  $y \in M$  and  $r < p_n$  such that  $(y + y + \cdots + y) + r = a$  ( $p_n$  ys). This search is guaranteed to terminate and both  $y$  and  $r$  are uniquely determined by  $n$  and  $a$ , by Euclidean division in PA. If  $r = 0$  we conclude  $n \in C$ , and  $n \notin C$  otherwise.  $\dashv$

It is natural to ask how far this argument can be pushed, replacing the theory PA by weaker theories. In the form that I have just given it, overspill is

only required for  $\Delta_0$  relations, and the subtheory  $I\Delta_0$  consisting of some base axioms and induction on  $\Delta_0$  formulas is strong enough to prove enough facts about Euclidean division, primes (including a formula for the  $n$ th prime for all standard and sufficiently small nonstandard  $n$ , though it is still not known if this theory proves the infinitude of primes) for the above argument to go through. This was observed by Cegielski *et al.* [2] and is essentially the proof given in my *Models of Peano Arithmetic* [7]. Note too that the subtheory mentioned earlier,  $\Pi_2 - \text{Th}(\text{PA})$  of the  $\Pi_2$  consequences of PA, contains  $I\Delta_0$ . Indeed all axioms of  $I\Delta_0$  are  $\Pi_1$  and also axioms of PA; in fact  $I\Delta_0$  is very much weaker than  $\Pi_2 - \text{Th}(\text{PA})$ .

A much sharper result using essentially the same ideas was achieved by Wilmers [29] who showed the same result for the subtheory  $IE_1$  of  $I\Delta_0$  where induction axioms are only allowed for bounded existential formulas, i.e. formulas of the form  $\exists \bar{y} < p(\bar{x}) q(\bar{x}, \bar{y}) = r(\bar{x}, \bar{y})$  where  $p, q, r$  are polynomials with nonnegative integer coefficients. Wilmers achieved some improvements on the above argument, firstly by taking  $A, B$  to be disjoint r.e. recursively inseparable sets of primes, but more particularly by using the MRDP theorem on the diophantine representation of r.e. sets [13]. Interestingly, Wilmers did not ever need to use the provability of the MRDP theorem, just its truth for the standard model  $\mathbb{N}$ , to get the required sets represented and to achieve the necessary overspill arguments in  $IE_1$ .

Two other methods for showing  $\text{SSy}(M)$  contains non-recursive sets come to mind. The first is to use a formalization of a (necessarily partial) truth definition in the model  $M$ . For example, there is a  $\Pi_1$  formula  $\text{Tr}_1(x)$  expressing “ $x$  is the Gödel-number of a true  $\Pi_1$  sentence”. This formula behaves as expected, and in particular is “correct” on standard formulas, in all models of PA. Then in such a model  $M$  the set  $C = \{\sigma \in \Pi_1 : M \models \text{Tr}_1(\sigma)\}$  of true standard  $\Pi_1$  sentences is coded, consistent, and (by the Gödel–Rosser theorem) therefore non-recursive. This method is straightforward and uses well-known results, but because as it relies on a formalization inside the theory of arithmetic, it is not applicable to weaker systems. No such truth definition is known for  $I\Delta_0$ , for instance.

The second method goes back to Scott [24], and uses overspill again. By an overspill argument,  $\text{SSy}(M)$  has the property that it is a boolean subalgebra of  $\mathcal{P}(\mathbb{N})$  closed under relative recursion and König’s lemma. (In modern terminology, it is a *Scott set*.) It follows that every first-order theory (such as PA itself) coded in  $M$  (i.e. in  $\text{SSy}(M)$ ) has a coded complete extension, and of course such sets will not themselves be recursive. (Details are given in *Models of Peano Arithmetic* [7].) This argument works very well in contexts where overspill is available, and Wilmers [29] shows that  $\text{SSy}(M)$  is a Scott set whenever  $M \models IE_1$  is nonstandard. For weaker theories we still seem to need alternative arguments.

In closing this section, I should mention two other strengthenings of Tennenbaum's theorem that have been considered. The first is to start with a nonstandard model  $M$  of some weak theory of arithmetic  $T$  and find a nonstandard initial segment  $I$  of  $M$  satisfying PA. Then Tennenbaum's theorem for the initial segment readily implies Tennenbaum's theorem for the original model  $M$  by the absoluteness of Euclidean division. For  $T = I\Delta_0$  a result of this kind was discovered by Cegielski *et al.* [2] and for  $T = IE_1$  nonstandard initial segments satisfying PA were discovered by Paris [19]. The other strengthening is to consider the reduct of the model  $M$  to addition alone. For  $M \models \text{PA}$ , this reduct is a model of Presburger arithmetic and is recursively saturated. Such models are necessarily nonrecursive by similar reasons: Euclidean division gives a notion of standard system, which (by recursive saturation) is a Scott set and therefore contains nonrecursive sets. The same result also holds for nonstandard  $M \models I\Delta_0$  (Cegielski *et al.* [2]) and for nonstandard  $M \models IE_1$  (Wilmer [29]).

**§3. Diophantine problems.** In the last section, we saw Tennenbaum's theorem, its relationship with the incompleteness theorems and some subsequent refinements of it down to the theory of bounded existential induction,  $IE_1$ . Some minor improvements were found by the present author by considering the parameter-free induction scheme for bounded existential formulas [6], but the theory  $IE_1$  represents the approximate limit of results of this kind to date. On the other hand Shepherdson [25] found nonstandard recursive models for quantifier-free induction, and Berarducci and Otero have improved this slightly to the case of normal open induction [1]. Between the theories  $IE_1$  and normal open induction there remains substantial work to be done.

These considerations already put us in the realm of studying diophantine sets, diophantine induction and diophantine definitions, and the issues relating to Hilbert's 10th problem. Wilmer used the MRDP theorem to prove his results about  $IE_1$ . It seems that further progress in the direction of Tennenbaum-type results will require new proofs or at least a more detailed analysis of existing proofs of the MRDP theorem.

**DEFINITION 3.1.** We say that a theory  $T$  of arithmetic *proves the MRDP theorem* ( $T \vdash \text{MRDP}$ ) if for every  $\Sigma_1$  formula of the language of arithmetic  $\varphi(\bar{x})$  there is an  $\exists_1$  formula  $\psi(\bar{x})$  in the same free variables such that

$$T \vdash \forall \bar{x} (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

The definition just given does not make any predetermined choices of equivalent formulas. (In this sense it is not about any first order axiom scheme being provable.) The theory  $I\Delta_0 + \text{exp}$  does prove MRDP (Gaifman and Dimitracopoulos [4]). On the other hand  $I\Delta_0 \vdash \text{MRDP}$  is an important open problem related to questions in complexity theory.

**THEOREM 3.2 (Kaye).** *Suppose  $T$  is a theory of arithmetic extending  $PA^-$  and  $T \vdash MRDP$ . Then every nonstandard model of  $T$  is nonrecursive.*

**PROOF.** Let  $A = \{n \in \mathbb{N} : \mathbb{N} \models \exists y \alpha(n, y)\}$  and  $B = \{n \in \mathbb{N} : \mathbb{N} \models \exists z \beta(n, y)\}$  be r.e. recursively inseparable sets, where  $\alpha, \beta$  are  $\Delta_0$ . We consider the formula  $\theta(w)$  which is

$$\forall x, y, z < w \neg (\alpha(x, y) \wedge \beta(x, z)).$$

The formula  $\theta(w)$  is  $\Delta_0$  and true for all  $w \in \mathbb{N}$ .

If there is some nonstandard  $c \in M$  such that  $M \models \theta(c)$  then

$$C = \{n \in \mathbb{N} : M \models \exists y < c \alpha(n, y)\}$$

would be  $\Delta_0$  definable, contains  $A$  (since  $c > \mathbb{N}$ ) and disjoint from  $B$ . So  $C$  would be nonrecursive, but, by considering both the formula defining  $C$  and its negation and  $MRDP$  in  $M$ , it is both  $\exists_1$ - and  $\forall_1$ -definable in  $M$ , hence is recursive in the operations  $+$ ,  $\cdot$ ,  $<$  of  $M$ . We conclude that at least one of these operations must be nonrecursive.

If on the other hand, the standard cut is defined in  $M$  by  $\theta(w)$ , i.e.  $\mathbb{N} = \{n \in \mathbb{N} : M \models \theta(n)\}$ , we take *any* nonstandard  $c \in M$ . Then a  $\Sigma_1$  predicate over naturals  $\exists x \psi(n, x)$  is true for  $n \in \mathbb{N}$  iff  $M \models \exists x < c (\theta(x) \wedge \psi(n, x))$ . This is a  $\Delta_0$  formula, so by  $MRDP$  in  $M$  it is both  $\exists_1$  and  $\forall_1$  definable in  $M$  so we have shown that every  $\Sigma_1$  predicate is recursive in the operations  $+$ ,  $\cdot$ ,  $<$  of  $M$ , and hence at least one of these operations must be nonrecursive.  $\neg$

The above result and its proof (which appeared first in Kaye [9]) are interesting, not so much in the result they prove, but in the way that they show that Tennenbaum-type theorems can be proved by means other than overspill, especially by reducing the problem in hand to questions about the standard model. These arguments seem to split into cases: one case mirrors the overspill argument, but if overspill fails then the standard cut is definable in the model in some way which also leads to the required conclusion. Other similar examples will be given shortly.

An old question that has been open since the 1950s (originally raised by Kreisel, I believe) is whether there is an algorithm for deciding provability in the system of arithmetic formulated in the logic without quantifiers. Shepherdson reformulated this question in terms of open induction, and it remains one of the most interesting questions about open induction.

**QUESTION 3.3.** Is the set  $\forall_1 - \text{Th}(\text{IOpen})$  of universal consequences of open induction recursive?

Another way of stating such a question is to ask if there is an effective method to decide, given a diophantine equation  $p(\bar{x}) = q(\bar{x})$  where  $p, q$  are polynomials with nonnegative integer coefficients, whether there is some model of  $\text{IOpen}$  containing at least one solution of the equation.

Questions of this type are related to Tennenbaum phenomena, for if  $T$  is a theory extending  $PA^-$  and  $\forall_1 - \text{Th}(T)$  is recursive then by forcing with quantifier-free conditions (as in the proof of Theorem 2.3 above) we may build a recursive model of  $\forall_2 - \text{Th}(T)$ . This model is nonstandard because the set of  $\exists_1$  sentences true in the model can be read off recursively from the construction, but the  $\exists_1$  theory of the standard model  $\mathbb{N}$  is not decidable by the MRDP theorem in  $\mathbb{N}$ . So Tennenbaum's theorem for  $\forall_2 - \text{Th}(T)$  would imply the undecidability of  $\forall_1 - \text{Th}(T)$ .

Now, although it is in fact true that  $\text{IOpen} = \forall_2 - \text{Th}(\text{IOpen})$ , the argument in the last paragraph doesn't seem to help us, as Tennenbaum's theorem fails for  $\text{IOpen}$ , by Shepherdson's work. In fact, this isn't quite the end of the story as we shall see in a moment, but let us for the moment set our sights a little lower and prove some weaker theorems of the same type. The following result has not previously been published.

**THEOREM 3.4.** *The set  $\forall\exists^< - \text{Th}(PA^-)$  of consequences of  $PA^-$  of the form  $\forall\bar{x}\exists\bar{y} < p(\bar{x}) \theta(\bar{x}, \bar{y})$  with  $\theta(\bar{x}, \bar{y})$  quantifier-free is not recursive.*

**PROOF.** This is a model theoretic forcing argument with  $\forall^<$  conditions, i.e. conditions of the form  $\forall\bar{y} < p(\bar{x}) \theta(\bar{x}, \bar{y})$  with  $\theta$  quantifier-free. (Note that up to logical equivalence such formulas are closed under conjunctions.) Our assumption that  $\forall\exists^< - \text{Th}(PA^-)$  is recursive means that we build a recursive model  $M$  of  $PA^-$  with a recursive sequence of conditions. For any tuple  $\bar{w} \in M$  of the constructed model, the set of formulas  $\exists\forall^< - \text{tp}(\bar{w})$  of all  $\exists\forall^<$  formulas true of the tuple  $\bar{w}$  is recursive since given such a formula  $\exists\bar{x}\forall\bar{y} < p(\bar{x}) \theta(\bar{w}, \bar{x}, \bar{y})$  we may decide its truth in  $M$  by searching simultaneously either for a conjunct of a condition of the form  $\forall\bar{y} < p(\bar{x}) \theta(\bar{w}, \bar{u}, \bar{y})$  or for a proof that

$$PA^- \vdash \forall\bar{u}, \bar{v}, \bar{w} (\lambda(\bar{u}, \bar{w}) \rightarrow \neg\forall\bar{y} < p(\bar{v}) \theta(\bar{w}, \bar{v}, \bar{y}))$$

for some condition  $\lambda(\bar{u}, \bar{w})$ . In particular, the model  $M$  is nonstandard as the set of formulas  $\exists\forall^< - \text{tp}(0)$  for the standard model  $\mathbb{N}$  is non-recursive.

Now take  $A, B$  r.e. and recursively inseparable. By the MRDP theorem in  $\mathbb{N}$  we may assume  $A = \{n \in \mathbb{N} : \exists\bar{y} \alpha(n, \bar{y})\}$  and  $B = \{n \in \mathbb{N} : \exists\bar{z} \beta(n, \bar{z})\}$  where  $\alpha, \beta$  are quantifier-free. Now consider

$$\forall x, \bar{y}, \bar{z} < c \neg (\alpha(x, \bar{y}) \wedge \beta(x, \bar{z})).$$

If there is no nonstandard  $c \in M$  satisfying this, then the standard cut  $\mathbb{N}$  is  $\forall^<$ -definable in  $M$  and any r.e. set  $X \subseteq \mathbb{N}$  would be recursive, since, by MRDP in  $\mathbb{N}$ ,  $X$  is  $\{n \in \mathbb{N} : \exists\bar{w} \xi(n, \bar{w})\}$  for some quantifier-free  $\xi$ , and hence

$$X = \left\{ n \in \mathbb{N} : \exists\bar{w} \left( \bigwedge_i \theta(w_i) \wedge \xi(n, \bar{w}) \right) \right\}$$

where  $\theta(x)$  is the  $\forall^<$  formula defining  $\mathbb{N}$  in  $M$ . The truth of this can be read off  $\exists\forall^< - \text{tp}(0)$ .

On the other hand, if there is some nonstandard  $c \in M$  such that

$$\forall x, \bar{y}, \bar{z} < c \neg (\alpha(x, \bar{y}) \wedge \beta(x, \bar{z})).$$

we consider  $C = \{n \in \mathbb{N} : \exists \bar{y} < c \alpha(n, \bar{y})\}$  which contains  $A$  and is disjoint from  $B$  by choice of  $C$ , and is also recursive since the truth of  $\exists \bar{y} < c \alpha(n, \bar{y})$  can be read off  $\exists \bar{y} < -\text{tp}(c)$  for any particular  $n$ .  $\neg$

We also have (Kaye [8]),

**THEOREM 3.5.** *The set  $\forall_1 - \text{Th}(IE_1)$  of  $\forall_1$  consequences of  $IE_1$  is not recursive.*

As will be clear from the proof of Theorem 3.4, there is still scope for stronger results here. I have given complex conditions by which induction axioms in a theory  $T$  may be omitted but nevertheless  $\forall_1 - \text{Th}(T)$  be proved nonrecursive by the methods discussed here [8]. Now I shall describe simpler conditions on the theory  $T$  for the same result to hold; one still strong enough to prove Theorem 3.5.

To motivate the ideas, consider first a model  $M \models I\Delta_0$ . The theory  $I\Delta_0$  is strong enough to define exponentiation in a reasonably clean way but cannot prove the statement of the totality of exponentiation,  $\text{exp}: \forall x \exists y y = 2^x$ . As already mentioned,  $I\Delta_0 + \text{exp}$  proves the MRDP theorem. The key application of MRDP in the arguments here is to the formula

$$\forall x, y, z < w \neg (\alpha(x, y) \wedge \beta(x, z))$$

where  $\alpha, \beta$  are quantifier-free. Let  $2_0^x = x$  and  $2_{k+1}^x = 2^{2_k^x}$ . By the provability of MRDP in  $I\Delta_0 + \text{exp}$  there is some  $k \in \mathbb{N}$  (depending possibly on  $\alpha, \beta$ ) and quantifier-free  $\theta(\bar{u})$  such that  $I\Delta_0$  proves

$$\forall w (\exists v (v = 2_k^w) \rightarrow ((\forall x, y, z < w \neg (\alpha(x, y) \wedge \beta(x, z))) \leftrightarrow \exists \bar{u} \theta(\bar{u}, w))).$$

This is by a result for  $I\Delta_0 + \text{exp}$  similar to the well-known theorem of Parikh [18] for  $I\Delta_0$ . Parikh's theorem says that if  $I\Delta_0 \vdash \forall \bar{x} \exists \bar{y} \theta(\bar{x}, \bar{y})$  where  $\theta(\bar{x}, \bar{y})$  is  $\Delta_0$  then the  $\bar{y}$  may be bounded by a polynomial in  $\bar{x}$ ,  $I\Delta_0 \vdash \forall \bar{x} \exists \bar{y} < t(\bar{x}) \theta(\bar{x}, \bar{y})$ . The same holds for  $I\Delta_0 + \text{exp}$  except the term  $t(\bar{x})$  must be a term involving exponentiation, and in fact the axiom  $\text{exp}$  can be omitted in the conclusion. (This is proved by a model-theoretic argument similar to the one for Parikh's Theorem given for example in *Models of Peano Arithmetic*, Exercise 6.5 [9].) The particular sentence of interest in our case is

$$\forall w (\forall x, y, z < w \neg (\alpha(x, y) \wedge \beta(x, z)) \rightarrow \exists \bar{u} \theta(\bar{u}, w))$$

which is provable in  $I\Delta_0 + \text{exp}$ , showing the  $\bar{u}$  can be bounded by an exponential term in  $w$ .

The idea then is to consider the subset  $L$  of  $M$  consisting of all  $w$  for which  $M \models \exists v (v = 2_k^w)$ . In the case of a nonstandard model of  $I\Delta_0$ , this  $L$  will be a nonstandard initial segment, but the important feature is that, by replacing

$\exists v (v = 2_k^w)$  by an appropriate Pell equation, we may take  $L$  to be definable by an existential formula.

DEFINITION 3.6. Let  $T$  be a theory in the usual language for arithmetic. We say that  $T$  *virtually proves MRDP* if whenever  $\varphi(w)$  is  $\forall^<$  then there are  $\psi(\bar{v}, w)$  and  $\theta(\bar{u}, w)$ , both quantifier-free, such that  $T \vdash \exists \bar{v} \psi(\bar{v}, n)$  for each  $n \in \mathbb{N}$  and

$$T \vdash \forall w (\exists \bar{v} \psi(\bar{v}, w) \rightarrow (\varphi(w) \leftrightarrow \exists \bar{u} \theta(\bar{u}, w))).$$

THEOREM 3.7. (a) *if  $T$  is consistent and virtually proves MRDP then  $\forall_1 - \text{Th}(T)$  is nonrecursive.*

(b)  *$IE_1$  virtually proves MRDP.*

PROOF. (a) Let  $A, B$  be r.e. and recursively inseparable and defined by  $A = \{n \in \mathbb{N} : \exists \bar{y} \alpha(n, \bar{y})\}$  and  $B = \{n \in \mathbb{N} : \exists \bar{z} \beta(n, \bar{z})\}$  as before, and let  $\varphi(w)$  be  $\forall x, \bar{y}, \bar{z} < w \neg (\alpha(x, \bar{y}) \wedge \beta(x, \bar{z}))$ . Take  $\psi(\bar{v}, w)$  and  $\theta(\bar{u}, w)$  as in the definition and build a recursive model  $M$  of  $\forall_2 - \text{Th}(T)$  in which all  $\exists_1 - \text{tp}(\bar{c})$  are resursive, as before. Now choose a nonstandard  $c \in M$  such that

$$M \models \exists \bar{u}, \bar{v} (\psi(\bar{v}, c) \wedge \theta(\bar{u}, c)).$$

Observe that each  $c \in \mathbb{N}$  satisfies the above formula, by properties of  $\psi$  and  $\theta$  in the definition, so if there were no nonstandard  $c \in M$  as above we would conclude that  $\mathbb{N}$  is  $\exists_1$ -definable in  $M$  and so  $\exists_1 - \text{Th}(\mathbb{N})$  can be read off  $\exists_1 - \text{tp}(0)$ , which is impossible.

Thus  $M \models \forall x, \bar{y}, \bar{z} < c \neg (\alpha(x, \bar{y}) \wedge \beta(x, \bar{z}))$  and the set

$$C = \{n \in \mathbb{N} : \exists \bar{y} < c \alpha(x, \bar{y})\}$$

is recursive and separates  $A, B$ .

(b) A Pell equation can be used in place of  $y = 2^x$  to prove an analogous result to

$$\forall w (\exists v (v = 2_k^w) \rightarrow ((\forall x, y, z < w \neg (\alpha(x, y) \wedge \beta(x, z))) \leftrightarrow (\exists \bar{u} \theta(\bar{u}, w))))).$$

for  $IE_1$  in place of  $ID_0$ . See Kaye [6] for details.  $\dashv$

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# HIERARCHIES OF SUBSYSTEMS OF WEAK ARITHMETIC

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**Abstract.** We completely characterize the logical hierarchy of various subsystems of weak arithmetic, namely:  $ZR$ ,  $ZR + N$ ,  $ZR + GCD$ ,  $ZR + Bez$ ,  $OI + N$ ,  $OI + GCD$ ,  $OI + Bez$ .

**§1. Introduction.** In 1964 Shepherdson [6] introduced a weak system of arithmetic, Open Induction ( $OI$ ), in which the Tennenbaum phenomenon does not hold. More precisely, if we restrict induction just to open formulas (with parameters), then we have a recursive nonstandard model. Since then several authors have studied Open Induction and its related fragments of arithmetic. For instance, since Open Induction is too weak to prove many true statements of number theory (It cannot even prove the irrationality of  $\sqrt{2}$ ), a number of algebraic first order properties have been suggested to be added to  $OI$  in order to obtain closer systems to number theory. These properties include: Normality [9] (abbreviated by  $N$ ), having the GCD property [8], being a Bezout domain [3, 8] (abbreviated by  $Bez$ ), and so on. We mention that GCD is stronger than  $N$ ,  $Bez$  is stronger than GCD and  $Bez$  is weaker than  $IE_1$  ( $IE_1$  is the fragment of arithmetic based on the induction scheme for bounded existential formulas and by a result of Wilmers [11], does not have a recursive nonstandard model). Boughattas in [1, 2] studied the non-finite axiomatizability problem and established several new results, including: (1)  $OI$  is not finitely axiomatizable, (2)  $OI + N$  is not finitely axiomatizable. To show that, he defined and considered the subsystems  $(OI)_p$  of  $(OI)$  and  $(N)_n$  of  $N$  ( $1 \leq p, n < \omega$ ) (See the next section for the definitions) and proved:

**THEOREM 1.1** (Boughattas [1]). (1)  $(OI)_p$  is finitely axiomatizable,  
(2) Suppose  $(p!, p') = 1$ , then  $(OI)_p \not\vdash (OI)_{p'}$ .

**THEOREM 1.2** (Boughattas [2], Theorem 2). Suppose  $(p!, p') = 1$  and  $(n!, n') = 1$ ,

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- (1)  $N + (OI)_p \not\vdash (OI)_{p'}$ ,
- (2)  $(N)_n + (OI) \not\vdash (N)_{n'}$ ,
- (3)  $(OI)_p + \neg(OI)_{p'} + (N)_n + \neg(N)_{n'}$  is consistent.

In [4] we strengthened Theorem 1.1 (2) to completely characterize the logical hierarchy of OI, by showing that  $(OI)_p \not\vdash (OI)_{p+1}$  iff  $p \neq 3$ . In this paper by modifying Boughattas' original proofs, we also strengthen Theorem 1.2 in two directions and completely characterize the logical hierarchy of OI + N, OI + GCD, OI + Bez:

**THEOREM C.**  $Bez + (OI)_p \not\vdash (OI)_{p+1}$ , when  $p \neq 3$ .

**THEOREM D.**  $(OI)_p + \neg(OI)_{p+1} + (N)_n + \neg(N)_{n+1}$  is consistent, when  $p \neq 3$ .

So we will have the following immediate consequences:

**COROLLARY E.** (1)  $N + (OI)_p \not\vdash (OI)_{p+1}$ , when  $p \neq 3$ .

(2)  $GCD + (OI)_p \not\vdash (OI)_{p+1}$ , when  $p \neq 3$ .

(3) All of the following subsystems of arithmetic are non-finite axiomatizable:  
 $OI$ ,  $OI + N$ ,  $OI + GCD$ ,  $OI + Bez$ ,  $(OI)_p + N$ ,  $OI + (N)_n$ .

In Theorems A and B of this paper, we consider the ZR versions of the above theorems. ZR is a subsystem of arithmetic that allows Euclidean division over each non-zero natural number  $n \in \mathbb{N}$ . ZR is introduced by Wilkie [10] in which he proved that ZR and OI have the same  $\forall_1$ -consequences. Later developments showed that ZR had very important role in constructing models of OI (See Macintyre-Marker [3], Smith [8]). ZR + N has also been studied in [5]. In Theorem A, we study natural subsystems  $(ZR)_S$  of (ZR), for a nonempty subset  $S$  of the set of prime numbers  $\mathbb{P}$  (see the next section for definition) and show that:

**THEOREM A.** Suppose  $S$  is a nonempty subset of  $\mathbb{P}$  and  $q$  is a prime number such that  $q \notin S$ , then  $(ZR)_S + Bez \not\vdash (ZR)_q$ .

Boughattas in ([2], Lemma 5) proved that DOR + N and ZR + N are not finitely axiomatizable. More precisely he showed that:

**THEOREM 1.3** (Boughattas [2], Lemma 5). Suppose  $(n!, n') = 1$ . Then  $ZR + (N)_n \not\vdash (N)_{n'}$ .

We modify Boughattas' proof and strengthen the above theorem in Theorem B:

**THEOREM B.** Suppose  $S$  is a nonempty subset of  $\mathbb{P}$  and  $q$  is a prime number such that  $q \notin S$ , then  $(ZR)_S + (N)_n + \neg(ZR)_q + \neg(N)_{n+1}$  is consistent.

Therefore we will have the following immediate implications:

**COROLLARY F.** Suppose  $S$  is a nonempty subset of  $\mathbb{P}$  and  $q$  is prime number such that  $q \notin S$ , then

- (1)  $(ZR)_S + N \not\vdash (ZR)_q$ .
- (2)  $(ZR)_S + GCD \not\vdash (ZR)_q$ .

- (3) *All of the following subsystems of arithmetic are non-finite axiomatizable:  $ZR, ZR + N, ZR + GCD, ZR + Bez, (ZR)_S + N, ZR + (N)_n$ , when  $S$  is an infinite subset of the set of prime numbers.*

**§2. Preliminaries.** Let  $L$  be the language of ordered rings based on the symbols  $+, -, \cdot, 0, 1, \leq$ . We write  $\mathbb{N}^*$  for  $\mathbb{N} \setminus \{0\}$ . We will work with the following set of axioms in  $L$ :

**DOR:** discretely ordered rings, i.e., axioms for ordered rings and

$$\forall x \neg(0 < x < 1).$$

**ZR:** discretely ordered  $\mathbb{Z}$ -rings, i.e., DOR and for every  $n \in \mathbb{N}^*$

$$\forall x \exists q, r \left( x = nq + r \bigwedge 0 \leq r < n \right).$$

We denote the sentence “DOR +  $\forall x \exists q, r (x = nq + r \bigwedge 0 \leq r < n)$ ” by  $(ZR)_n$ . Suppose  $\mathbb{P}$  denote the set of prime numbers of  $\mathbb{N}$ . Let  $S$  be a nonempty subset of  $\mathbb{P}$ . We define the subsystem  $(ZR)_S$  of ZR as the below:

**$ZR_S$ :** DOR + for every  $p \in S$

$$\forall x \exists q, r \left( x = qp + r \bigwedge 0 \leq r < p \right).$$

If  $S = \{p_{i_1}, \dots, p_{i_n}\}$  is a finite subset of  $\mathbb{P}$ , we write  $(ZR)_{p_{i_1}, \dots, p_{i_n}}$  instead of  $(ZR)_{\{p_{i_1}, \dots, p_{i_n}\}}$ . This is consistent with the above notation  $(ZR)_n$ .

**OI:** open induction, i.e., DOR and for every open  $L$ -formula  $\psi(\bar{x}, y)$

$$\forall \bar{x} \left( \psi(\bar{x}, 0) \bigwedge \forall y \geq 0 (\psi(\bar{x}, y) \rightarrow \psi(\bar{x}, y + 1)) \rightarrow \forall y \geq 0 \psi(\bar{x}, y) \right).$$

By considering the fact that in discretely ordered rings an open  $L$ -formula  $\varphi(\bar{x}, y)$  can be written as a Boolean combination of polynomial equalities and inequalities with the variable  $y$  and the parameters  $\bar{x}$ , there exist natural numbers  $m, n$  such that:

$$\varphi(\bar{x}, y) = \bigwedge_{i \leq m} \bigvee_{j \leq n} p_{ij}(\bar{x}, y) \leq q_{ij}(\bar{x}, y),$$

we can define the degree of  $\varphi(\bar{x}, y)$  relative to  $y$  by

$$\deg \varphi(\bar{x}, y) = \max\{\deg_y p_{ij}(\bar{x}, y), \deg_y q_{ij}(\bar{x}, y) | i \leq m, j \leq n\}.$$

**$(OI)_p$ :** open induction up to degree  $p$  (i.e., DOR and for every open  $L$ -formula  $\psi(\bar{x}, y)$  with  $\deg \psi(\bar{x}, y) \leq p$

$$\forall \bar{x} \left( \psi(\bar{x}, 0) \bigwedge \forall y \geq 0 (\psi(\bar{x}, y) \rightarrow \psi(\bar{x}, y + 1)) \rightarrow \forall y \geq 0 \psi(\bar{x}, y) \right)).$$

**N**: normality (i.e., being domain and integrally closed in its fraction field, namely for every  $n \in \mathbb{N}^*$ ,  $\forall x, y, z_1, \dots, z_n$

$$(y \neq 0 \wedge x^n + z_1 x^{n-1} y + \dots + z_{n-1} x y^{n-1} + z_n y^n = 0 \longrightarrow \exists z (yz = x))).$$

**(N)<sub>n</sub>**: normality up to degree  $n \in \mathbb{N}^*$  (i.e., being domain and for every  $m \in \mathbb{N}^*$ ,  $m \leq n$ ,  $\forall x, y, z_1, \dots, z_m$

$$(y \neq 0 \wedge x^m + z_1 x^{m-1} y + \dots + z_{m-1} x y^{m-1} + z_m y^m = 0 \longrightarrow \exists z (yz = x))).$$

It is clear that any domain satisfies  $(N)_1$ .

**GCD**: having greatest common divisor (i.e., the usual axioms for being a domain plus

$$\forall x, y (x = y = 0 \vee \exists z (z|x \wedge z|y \wedge (\forall t ((t|x \wedge t|y) \rightarrow t|z)))).$$

where  $x|y$  is an abbreviation for  $\exists t (t \cdot x = y)$ ).

**Bez**: the usual axioms for being a domain plus the *Bezout property*:

$$\forall x, y \exists z, t ((xz + yt)|x \wedge (xz + yt)|y),$$

namely, every finitely generated ideal is principal.

It is known that  $\text{Bez} \vdash \text{GCD} \vdash \text{N}$ , and  $\text{OI} \not\vdash \text{OI} + \text{N} \not\vdash \text{OI} + \text{GCD} \not\vdash \text{OI} + \text{Bez}$  (Smith [7], Lemmas 1.9 and 1.10).

Also we will need another algebraic property, though it is not first-order expressible:

**DCC**: let  $M$  be a domain.  $M$  has the *divisor chain condition* (DCC) if  $M$  contains no infinite sequence of elements  $a_0, a_1, a_2, \dots$  such that each  $a_{i+1}$  is a proper divisor of  $a_i$  (i.e.,  $a_i/a_{i+1}$  is a nonunit).

Let  $M$  be an ordered domain (resp. a domain), then  $RC(M)$  (resp.  $AC(M)$ ) will denote the real closure (resp. the algebraic closure) of its fraction field. It is well known that  $AC(M) = RC(M)[\sqrt{-1}]$ . Let  $p \in \mathbb{N}^*$  and  $F$  be an ordered field (resp. a field), we define the  $p$ -real closure (resp. the  $p$ -algebraic closure) of  $F$ , denoted by  $RC_p(F)$  (resp.  $AC_p(F)$ ), to be the smallest subfield of  $RC(F)$  (resp.  $AC(F)$ ) containing  $F$  such that every polynomial of degree  $\leq p$  with coefficients in  $RC_p(M)$  (resp.  $AC_p(F)$ ) which has a root in  $RC(F)$  (resp.  $AC(F)$ ) also has a root in  $RC_p(F)$  (resp.  $AC_p(F)$ ). Similarly if  $M$  be an ordered domain (resp. a domain), then  $RC_p(M)$  (resp.  $AC_p(M)$ ) will denote the  $p$ -real closure (resp. the  $p$ -algebraic closure) of its fraction field.

It can be shown that  $AC_p(M) = RC_p(M)[\sqrt{-1}]$ . Similar to real closed fields and algebraic closed fields, it is also easily seen that:

(1) *If  $P(x)$  is a polynomial of degree  $\leq p$  with the coefficients in  $RC_p(F)$  and  $P(a) < 0 < P(b)$ , for some  $a < b$  in  $RC_p(F)$ , then there exists a  $c \in RC_p(F)$ , such that  $a < c < b$  and  $P(c) = 0$ .*

(2) *If  $P(x)$  is a polynomial of degree  $\leq p$  with the coefficients in  $AC_p(F)$ , then  $P(x)$  can be represented as a product of linear factors with coefficients in  $AC_p(F)$ .*

Properties (1) and (2) can define and axiomatize the notions of *p-real closed field* and *p-algebraic closed field*, denoted by  $(RCF)_p$  and  $(ACF)_p$ , respectively.

Given two ordered domains  $I \subset K$  we say that  $I$  is an *integer part* of  $K$  if  $I$  is discrete and for every element  $\alpha \in K$ , there exists an element  $a \in I$  such that  $0 \leq \alpha - a < 1$ . We call  $a$ , the *integer part* of  $\alpha$ , and sometimes denote it by  $[\alpha]_I$ . Shepherdson and Boughattas characterized models of  $(OI)_p$ , in terms of *p*-real closed fields ( $1 \leq p \leq \omega$ ):

**THEOREM 2.1** (Shepherdson [6]). *Let  $M$  be an ordered domain.  $M$  is a model of  $OI$  iff  $M$  is an integer part of  $RC(M)$ .*

**THEOREM 2.2** (Boughattas [1, 2]). *Let  $M$  be an ordered domain.  $M$  is a model of  $(OI)_p$  iff  $M$  is an integer part of  $RC_p(M)$ .*

We also need a fact from Puiseux series:

**DEFINITION 2.3.** Let  $K$  be a field. The following is the field of Puiseux series in descending powers of  $x$  with coefficients in  $K$ :

$$K((x^{1/\mathbb{N}})) = \left\{ \sum_{k \leq m} a_k x^{k/r} : m \in \mathbb{Z}, r \in \mathbb{N}^*, a_k \in K \right\}.$$

**THEOREM 2.4** (Boughattas [1]). *( $1 \leq p \leq \omega$ )  $K$  is a *p*-real (resp. *p*-algebraically) closed field iff  $K((x^{1/\mathbb{N}}))$  is a *p*-real (resp. *p*-algebraically) closed field.*

### §3. The main results.

**3.1. Proof of Theorem A.** Suppose  $S$  is a subset of the set of prime numbers  $\mathbb{P}$ . We present here a *relative to  $S$*  version of some theorems of (Smith [8]) that is needed for proving theorem A. Interestingly, all proofs of (Smith [8]) remain valid, if we make routine changes which will be explained. We mention that when  $S = \mathbb{P}$ , we get the original definitions and theorems. We first define  $\widehat{\mathbb{Z}}_S = \prod_{p \in S} \mathbb{Z}_p$ , where  $\mathbb{Z}_p$  is the ring of *p*-adic integers, and  $\langle S \rangle = \{p_1^{\alpha_1} \cdots p_n^{\alpha_n}; n \in \mathbb{N}^*, \alpha_i \in \mathbb{N} \text{ and } p_i \in S\}$ . It is clear that there is the canonical embedding of  $\langle S \rangle$  in  $\widehat{\mathbb{Z}}_S$ .

Let  $M$  be a model of  $(\text{ZR})_S$ , by relativizing to  $S$ , we get a (unique)  $S$  – remainder homomorphism  $\text{Rem}: M \longrightarrow \widehat{\mathbb{Z}}_S$  given by the projective limit of the canonical homomorphism

$$\psi_n : M \longrightarrow M/nM \cong \mathbb{Z}/n\mathbb{Z}$$

for  $n \in \langle S \rangle$ . See (Macintyre-Marker [3], Lemma 1.3).

Now we give the  $S$ -relativization of the so called  $\widehat{\mathbb{Z}}$ -construction. Let  $M$  be a discretely ordered ring with  $\varphi : M \longrightarrow \widehat{\mathbb{Z}}_S$  a homomorphism and assume that all standard primes remain prime in  $M$ . We form a new ring  $M_{\varphi,S} = \{a/n; a \in M, n \in \langle S \rangle \text{ and } n|\varphi(a) \text{ in } \widehat{\mathbb{Z}}_S\}$ . We extend  $\varphi$  to  $M_{\varphi,S}$  in the obvious way. We say that  $M_{\varphi,S}$  is obtained from  $M$  by the  $\widehat{\mathbb{Z}}_S$  – construction. By relativizing the proof of (Macintyre-Marker [3], Lemma 3.1) we get:

LEMMA 3.1.  $M_{\varphi,S} \models (\text{ZR})_S$ .

Parsimony of homomorphisms plays a very important role in Smith's constructions. Therefore we have the following definition:

DEFINITION 3.2. Let  $M$  be a discretely ordered ring with  $\varphi : M \longrightarrow \widehat{\mathbb{Z}}_S$  a homomorphism, where  $\varphi$  is the projective limit of the homomorphism  $\psi_n : M \longrightarrow \mathbb{Z}/n\mathbb{Z}$  for  $n \in \langle S \rangle$ . We say that  $\varphi$  is  $S$ -parsimonious if for each nonzero  $a \in M$  there are only finitely many  $n \in \langle S \rangle$  such that  $\psi_n(a) = 0$ .

The following lemma asserts that the  $\widehat{\mathbb{Z}}_S$ -construction preserves parsimony.

LEMMA 3.3. If  $\varphi : M \longrightarrow \widehat{\mathbb{Z}}_S$  is  $S$ -parsimonious, then the extension of  $\varphi$  to  $M_{\varphi,S}$  is  $S$ -parsimonious.

PROOF. The proof is the  $S$ -relativization of Smith's proof of Lemma 5.1. in [8]. Let  $0 \neq a/n \in M_{\varphi,S}$ , where  $a \in M$ ,  $n \in \langle S \rangle$ . Suppose  $\psi_k(a/n) = 0$ , for a  $k \in \langle S \rangle$ . Since  $M_{\varphi,S}$  is a model of  $(\text{ZR})_S$ , we have  $k|a/n$  in  $M_{\varphi,S}$ , so in particular  $k|a$  in  $M_{\varphi,S}$ . Thus  $\psi_k(a) = 0$ . Since  $\varphi : M \longrightarrow \widehat{\mathbb{Z}}_S$  is  $S$ -parsimonious, there are only finitely many possibilities for  $k \in \langle S \rangle$ .  $\dashv$

The following theorem says that in the presence of having a  $S$ -parsimonious map the  $\widehat{\mathbb{Z}}_S$ -construction preserves GCD and DCC.

THEOREM 3.4. Let  $M$  be a discretely ordered ring with the GCD (DCC). Let  $\varphi : M \longrightarrow \widehat{\mathbb{Z}}_S$  be  $S$ -parsimonious and in the DCC case the standard primes remain prime in  $M$ . Then  $M_{\varphi,S}$  has the GCD (DCC).

PROOF. We leave the proof to the reader as an easy and instructive exercise to adopt Smith's proofs of Theorems 5.3. and 5.5. in [8]. Just replace everywhere in the proof,  $\mathbb{Z}$ -ring by a model of  $(\text{ZR})_S$ ,  $\varphi : M \longrightarrow \widehat{\mathbb{Z}}$  by  $\varphi : M \longrightarrow \widehat{\mathbb{Z}}_S$ , parsimonious by  $S$ -parsimonious,  $M_{\varphi}$  by  $M_{\varphi,S}$ , and check that the arguments remain valid!  $\dashv$

Transcendental extensions preserve GCD and DCC.

**THEOREM 3.5** (Smith [8], Theorems 6.8. and 6.10). *Let  $M$  be a GCD (DCC) domain and suppose  $x$  is transcendental over  $M$ . Then  $M[x]$  is a GCD (DCC) domain.*

By the same adaptation of Theorem 6.12. of (Smith [8]), we see that  $S$ -parsimonious maps can be extended to transcendental extensions. More precisely:

**THEOREM 3.6.** *Let  $M$  be a countable model of  $(\mathbb{Z}R)_S$  and suppose the remainder homomorphism  $\varphi : M \longrightarrow \widehat{\mathbb{Z}}_S$  is  $S$ -parsimonious. Let  $x$  be transcendental over  $M$  and suppose  $M[x]$  is discretely ordered (and this ordering restricts to the original ordering on  $M$ ). Then  $\varphi$  can be extended to a  $S$ -parsimonious  $\varphi : M[x] \longrightarrow \widehat{\mathbb{Z}}_S$ , such that  $\varphi(x)$  is a unit of  $\widehat{\mathbb{Z}}_S$ .*

We will need in this paper to consider the property of *factoriality* (a factorial domain has the property that any nonunit has a factorization into irreducible elements, and this factorization is unique up to units). We will use the following theorem:

**THEOREM 3.7** (Smith [8], Theorem 1.5).  *$M$  is factorial iff  $M$  has both of the GCD property and DCC.*

In order to gain a Bezout domain the F-construction in Macintyre-Marker paper [3] has a crucial role. By combining Theorems 8.5 and 8.7 from (Smith [8]), Lemma 3.26 of (Macintyre-Marker [3]) and its proof, we have:

**THEOREM 3.8.** *Let  $M$  be a discretely ordered domain with DCC (GCD) and suppose  $v, w \in M$  are primes and  $x$  is larger than any element of  $M$ . Let  $M^* = M[x, \frac{1-xv}{w}]$ . Then  $M^*$  is a discretely ordered domain with DCC (GCD).*

In the following theorem we see that  $S$ -parsimony can be extended in F-constructions:

**THEOREM 3.9.** *Let  $M$  be a countable model of  $(\mathbb{Z}R)_S$  and the remainder homomorphism  $\varphi : M \longrightarrow \widehat{\mathbb{Z}}_S$  is  $S$ -parsimonious. Let  $v, w \in M$  be primes of  $M$  and  $w$  is nonstandard. Suppose  $x$  is transcendental over  $M$ , and the discrete ordering of  $M$  extends to discrete ordering on  $M^* = M[x, \frac{1-xv}{w}]$ . Then  $\varphi$  can be extended to  $S$ -parsimonious  $\varphi : M^* \longrightarrow \widehat{\mathbb{Z}}_S$ , such that  $\varphi(x)$  is a unit of  $\widehat{\mathbb{Z}}_S$ .*

**PROOF.** See the proof of Theorem 8.9. of (Smith [8]). ◀

The next theorem guarantees the preservation of the GCD property and DCC in chains constructed by alternative applications of the F-construction and the  $\widehat{\mathbb{Z}}_S$ -construction via parsimonious maps. We express the theorems in a more restricted and more suitable form which is adequate for us:

**THEOREM 3.10.** *Suppose  $M_0$  is a (GCD) DCC countable model of  $(\mathbb{Z}R)_S$  and there is  $S$ -parsimonious  $\varphi : M_0 \longrightarrow \widehat{\mathbb{Z}}_S$ . Let  $\{M_i : i \in \mathbb{N}\}$  be a chain of discretely ordered domains such that  $M_{2i+1}$  is constructed from  $M_{2i}$  by the  $\widehat{\mathbb{Z}}_S$ -construction, and  $M_{2i+2}$  is constructed from  $M_{2i+1}$  by the F-construction. In addition we suppose that in the DCC case, in the whole process of extending rings*

at most finitely many irreducibles have been killed (this means that only finitely many irreducibles will become reducible in later stages). Then  $M = \bigcup_{i \in \mathbb{N}} M_i$  is a model of (GCD) DCC.

PROOF. See Theorems 9.4. and 9.8. in (Smith [8]). ⊢

By the following series of easy lemmas, we will not worry about DCC in our chain of models in the proof of Theorem A:

LEMMA 3.11 (Smith [8], Lemma 3.8). *Let  $M$  be a GCD domain. Then  $p \in M$  is irreducible iff it is prime.*

Of course the following lemma needs an easy  $S$ -adaptation of Lemma 3.2 in (Macintyre-Marker [3]):

LEMMA 3.12. *Let  $M \models \text{DOR}$  and  $\varphi : M \longrightarrow \widehat{\mathbb{Z}}_S$  be a ring homomorphism and assume that all standard primes remain prime in  $M$ . If  $q \in M$  is irreducible and  $\varphi(q)$  is unit in  $\widehat{\mathbb{Z}}_S$ , then  $q$  is irreducible in  $M_{\varphi, S}$ .*

LEMMA 3.13 (Macintyre-Marker [3], Lemma 3.27). *If  $q$  is irreducible in  $M$ , then  $q$  is irreducible in  $M^*$ , constructed in Theorem 3.8 (by the  $F$ -construction).*

Now we have gathered all preliminaries to prove Theorem A:

THEOREM A. *Suppose  $S$  is a nonempty subset of  $\mathbb{P}$  and  $q$  is prime number such that  $q \notin S$ , then  $(\text{ZR})_S + \text{Bez} \not\vdash (\text{ZR})_q$ .*

PROOF. We do a suitable and modified version of Smith's process to construct a Bezout model of open induction (Smith [8] Theorem 10.7). We shall inductively construct an  $\omega$ -chain of models  $M_i$  such that  $\bigcup_i M_i = M_\omega$  will be a model of  $(\text{ZR})_S + \text{Bez} + \neg(\text{ZR})_q$ . We work inside the ordered field  $\mathbb{Q}(x_1, \dots, x_i, \dots)$  so that for each  $i \in \omega$ ,  $x_{i+1}$  is larger than any element of  $\mathbb{Q}(x_1, \dots, x_i)$  and  $x_1$  is infinitely large. We will do the  $F$ -construction at odd stages and the  $\widehat{\mathbb{Z}}_S$ -construction at even stages.

Take  $M_0 = \mathbb{Z}$  together the natural remainder  $S$ -parsimonious homomorphism  $\varphi : M_0 \longrightarrow \widehat{\mathbb{Z}}_S$ . Let us show what we do at stages  $2k + 1$ . Suppose  $M_{2k}$  and a  $S$ -parsimonious map  $\varphi : M_{2k} \longrightarrow \widehat{\mathbb{Z}}_S$ , have been constructed. At this stage we consider a pair of distinct primes  $v$  and  $w$  belonging to  $M_{2k}$  such that  $w$  is nonstandard. (Of course we do this in such a way that every such pair of primes in  $M_\omega$  will have been considered at some stage  $2k + 1$ ). Thus  $(v, w) = 1$  in  $M_{2k}$ . We define  $M_{2k+1} = M_{2k}[x_k, \frac{1-x_kv}{w}]$  according to Theorem 3.8. Suppose  $y_k = \frac{1-x_kv}{w}$ , then we have  $x_kv + y_kw = 1$  in  $M_{2k+1}$ . So  $(v, w)_B = 1$  in  $M_{2k+1}$ .  $((v, w)_B$  is the Bezout greatest common divisor of  $v$  and  $w$ , it means that  $(v, w)_B | v$  and  $(v, w)_B | w$  and there exist  $r$  and  $s$  in  $M_{2k+1}$  such that  $rv + su = 1$ ). We refer to (Smith [8], Section 3) for the basic related definitions and theorems. By Theorem 3.9  $\varphi$  is extended to a  $S$ -parsimonious map  $\varphi : M_{2k+1} \longrightarrow \widehat{\mathbb{Z}}_S$ . At stage  $2k + 2$ , we employ Lemma 3.1 and define

$M_{2k+2} = (M_{2k+1})_{\varphi, S}$  which is a model of  $(ZR)_S$ . Lemma 3.3 gives us the desired parsimonious extensions  $\varphi : M_{2k+2} \rightarrow \widehat{\mathbb{Z}}_S$ . Since  $(ZR)_S$  is a  $\forall\exists$ -theory, then it is preserved in chains, therefore  $M_\omega \models (ZR)_S$ .

Now we show that  $M_\omega$  is a Bezout domain. The proof is similar to (Smith [8], Theorem 10.7) with a minor change. By Theorems 3.4 and 3.8, each  $M_i$  has the GCD and DCC, so by Theorem 3.10  $M_\omega$  has both the GCD and DCC (by Lemmas 3.11, 3.12 and 3.13 we know that no irreducible is killed) and from Theorem 3.7 we conclude that  $M_\omega$  is a factorial domain. In order to show that  $M_\omega$  is a Bezout domain, by considering the fact that  $M_\omega$  has the GCD property, it suffices to prove that any two elements of  $M_\omega$  has the Bezout greatest common divisor. Let  $a, b \in M_\omega$  and let  $c = (a, b)$  in  $M_\omega$ . We can assume  $a, b > 1$ . Let  $a = a'c, b = b'c$  in  $M_\omega$ . So  $(a', b') = 1$  in  $M_\omega$ . Since  $M_\omega$  is factorial, we can write  $a' = mP_1^{e_1} \dots P_k^{e_k}$  and  $b' = nQ_1^{f_1} \dots Q_l^{f_l}$ , where  $m, n \in \mathbb{N}$  are nonzero,  $k, l \geq 0$  and the  $P_i, Q_j$  are nonstandard primes such that  $P_i \neq Q_j$  for all  $i, j$ . We will show that  $(a', b')_B = 1$ . Clearly  $(m, n)_B = 1$ . Suppose  $m = g_1^{v_1} \dots g_r^{v_r}$  and  $n = h_1^{w_1} \dots h_s^{w_s}$  are the prime factorizations of  $m, n$  in  $\mathbb{N}$ . By the F-construction every one of  $(P_i, g_j)_B = 1$ ,  $(Q_i, h_j)_B = 1$  and  $(P_i, Q_j)_B = 1$ , occur at some odd stage of our construction. Therefore by iterated applications of (Smith [8], Lemma 3.4), we conclude that  $(a', b')_B = 1$ . By (Smith [8], Lemma 3.4), we have  $c = (a, b)_B$  at some odd stage and then using (Smith [8], Lemma 3.7) we ensure that  $c = (a, b)_B$  in  $M_\omega$ . This completes the proof of the Bezoutness of  $M_\omega$ .

Note that in the original proof of Smith ([8], Theorem 10.7) he just considers pairs of nonstandard primes and doesn't need to consider pairs of primes such that one is standard and the other is nonstandard. Since his chain of domains are ZR-rings, this gives automatically the Bezout greatest common divisor for such pairs. But as we want ZR to fail in our model, we are forced to consider pairs of standard and nonstandard primes in the F-construction, as well.

Now we show that  $(ZR)_q$  fails in  $M_\omega$ . We first observe that in the first step of our construction, namely, when passing from  $M_0 = \mathbb{Z}$  to  $M_1$ , there is no nonstandard prime in  $M_0$ . So  $M_1$  is just  $\mathbb{Z}[x_1]$  and we have no  $y_1$ . On the other hand from the construction it is evident that elements of  $M_\omega$  are of the form  $f(x_1, x_2, y_2, \dots, x_k, y_k)$ , for some  $k$ , where  $f$  is a polynomial with the coefficients in the set  $\mathbb{Z}_{\langle S \rangle} = \{a/k; a \in \mathbb{Z} \text{ and } k \in \langle S \rangle\}$ . Now for a contradiction, suppose  $M_\omega$  is a model of  $(ZR)_q$ . Then there is a  $b \in M_\omega$  such that  $x_1 = bq + r$  with  $0 \leq r < q$ . Take  $b = f(x_1, x_2, y_2, \dots, x_k, y_k)$ , so we have  $x_1 = f(x_1, x_2, y_2, \dots, x_k, y_k)q + r$ . Observe that  $x_2, y_2, \dots, x_k, y_k$  are transcendental over  $\mathbb{Q}(x_1)$ , then  $f$  does not depend on them, so we can assume  $x_1 = f(x_1)q + r$ . Since  $x_1$  is also transcendental over  $\mathbb{Q}$ , it follow that the degree of  $f$  must be one. Thus  $f(x_1) = ax_1$  and  $a \in \mathbb{Z}_{\langle S \rangle}$ . So  $x_1 = ax_1q + r$  and then  $x_1(1 - aq) = r$ , which implies that  $a = 1/q$  and this is in contradiction with  $a \in \mathbb{Z}_{\langle S \rangle}$ , since  $q \notin \langle S \rangle$ .  $\dashv$

### 3.2. Proof of Theorem B.

**THEOREM B.** *Suppose  $S$  is a nonempty subset of  $\mathbb{P}$  and  $q$  is a prime number such that  $q \notin S$ , then  $(ZR)_S + (N)_n + \neg(ZR)_q + \neg(N)_{n+1}$  is consistent.*

**PROOF.** In [4] we proved that if  $n \neq 3$ , there is a  $\lambda$  which is real algebraic of degree  $n + 1$  over  $\mathbb{Q}$  and doesn't belong to  $RC_n(\mathbb{Q})$ . Now suppose  $x$  is an infinitely large element. For  $n \neq 3$ , fix  $\lambda$  as above. For  $n = 3$  we choose  $\lambda$  as a root of an irreducible polynomial of degree 4 such that  $\lambda \notin RC_2(\mathbb{Q})$ . Let  $A$  be the ring of integers of the algebraic number field  $\mathbb{Q}(\lambda)$ . Form  $A_{\langle S \rangle} = \{a/k; a \in A \text{ and } k \in \langle S \rangle\}$ . It is an elementary fact from algebraic number theory that  $A$  is a normal ring. Since  $A_{\langle S \rangle}$  is a localization of  $A$  relative to a multiplicative set, then it is also normal. Let  $M = \mathbb{Z}[rx; r \in A_{\langle S \rangle}]$ . We claim that  $M$  witnesses Theorem B. It is obvious that  $M \models (ZR)_S$ . By an argument similar to the last paragraph of the proof of theorem A, it is easily shown that  $M \models \neg(ZR)_q$ .

Now we prove  $M \models \neg(N)_{n+1}$ . Let  $v \in \mathbb{N}$  be such that  $v\lambda$  is an algebraic integer. Suppose  $P(t) \in \mathbb{Z}$  is its minimal polynomial of degree  $n + 1$  which is monic. Obviously  $v\lambda x \in M$ . But we have  $P(v\lambda x/x) = 0$ , while  $v\lambda \notin M$ . So  $M$  is not a model of  $(N)_{n+1}$ .

It remains to show that  $M \models (N)_n$ . Let  $u, v$  be nonzero elements of  $M$  such that

$$(u/v)^s + z_1(u/v)^{s-1} + \cdots + z_s = 0 \quad (z_1, \dots, z_s \in M, s \leq n).$$

We will show that  $u/v \in M$ . Notice that elements of  $M$  are those elements of  $A_{\langle S \rangle}[x]$  with integer constant coefficient.  $A_{\langle S \rangle}$  is normal, so is  $A_{\langle S \rangle}[x]$ . Thus  $u/v \in A_{\langle S \rangle}[x]$ . On the other hand, since  $\mathbb{Q}(\lambda)[x]$  is a factorial ring,  $u/v$  can be written as:

$$u/v = \rho \prod_{i \in I} P_i \prod_{j \in J} Q_j,$$

in which  $\rho \in \mathbb{Q}(\lambda)$ , the  $P_i$ 's are irreducible in  $\mathbb{Q}(\lambda)[x]$ , without constant coefficient and  $Q_j$ 's are irreducible in  $\mathbb{Q}(\lambda)[x]$  with the constant coefficient one. If  $I$  is nonempty, then  $\rho \prod_{i \in I} P_i \prod_{j \in J} Q_j$  has no constant coefficient and thus  $u/v \notin M$ . Now suppose  $I = \emptyset$ . Put  $x = 0$  in  $u, v, z_1, \dots, z_s$ . Therefore  $\rho$  is an algebraic integer with the degree, equal or less than  $n$  over  $\mathbb{Z}$ . We show it is one. If  $n = 1$  there is nothing to prove. If not, we have  $[\mathbb{Q}(\lambda) : \mathbb{Q}(\rho)] < n + 1$ . But

$$[\mathbb{Q}(\lambda) : \mathbb{Q}(\rho)][\mathbb{Q}(\rho) : \mathbb{Q}] = [\mathbb{Q}(\lambda) : \mathbb{Q}] = n + 1.$$

Then  $[\mathbb{Q}(\lambda) : \mathbb{Q}(\rho)]$  divides  $[\mathbb{Q}(\lambda) : \mathbb{Q}] = n + 1$ . So we have a chain of field extensions,  $\mathbb{Q} \subset \mathbb{Q}(\rho) \subset \mathbb{Q}(\lambda)$  such that  $[\mathbb{Q}(\lambda) : \mathbb{Q}(\rho)] \leq n - 1$  and  $[\mathbb{Q}(\rho) : \mathbb{Q}] \leq n - 1$ . This implies that  $\lambda \in RC_{n-1}(\mathbb{Q})$  which is in contradiction with the choice of  $\lambda$ . Hence  $\rho$  is an algebraic integer of degree one. So  $\rho \in \mathbb{Z}$

and this implies that  $u/v \in M$ , which means that  $M$  is model of  $(N)_n$ . This completes the proof of Theorem B.  $\dashv$

**3.3. Proofs of Theorems C and D.** In order to demonstrate Theorem C, we need a generalization of a theorem of Boughattas. In ([2], Theorem V.1.) Boughattas proved that every saturated ordered field admits a normal integer part. But we show that:

LEMMA 3.14. *Every  $\omega_1$ -saturated ordered field admits a Bezout integer part.*

PROOF. (*Sketch*) Suppose  $K$  is an  $\omega_1$ -saturated ordered field. Boughattas [2] in a series of three Lemmas: *Principal*, *Integer Part* and *Construction*, showed that we can build an  $\omega_1$ -chain of countable discretely ordered rings  $M_i, i < \omega_1$  such that  $M = \bigcup_{i < \omega_1} M_i$  is an integer part of  $K$ . Furthermore he considers an arbitrary subset  $\Lambda \subset K$  of real algebraic elements which plays a role in the construction of the  $M_i$ 's. Varying  $\Lambda$  gives us various kinds of integer parts. When  $\Lambda = \emptyset$ , we obtain a normal integer part and it is implicit in the paper that in this case, the  $M_i$ 's are obtained by alternative applications of the Wilkie-construction and the  $\widehat{\mathbb{Z}}$ -construction. But it must be noticed that even in this case the procedure of doing the  $\widehat{\mathbb{Z}}$ -construction is different from the original one, because it is no longer assumed that the ground field is dense in its real closure. To gain a Bezout integer part, we observe that we can do the procedure of the Theorems 10.7 and 10.8 of Smith [8] inside  $K$ . In this procedure we need the extra F-construction. Since any  $M_i, i < \omega_1$  is countable and  $K$  is  $\omega_1$ -saturated, then there is always an element  $b_i$  in  $K$  which is larger than any element of  $M_i$ . By Lemma 3.26 of (Macintyre-Marker [3]) we are sure that we can do the F-construction. To obtain an integer part of  $K$ , suppose  $(b_\alpha, \alpha < \omega_1)$  be an enumeration of elements of  $K$ . Let  $M_i$  has been constructed and at step  $i + 1$  we want to do the Wilkie-construction. We seek the least ordinal  $\alpha_i$ , such that  $b_{\alpha_i}$  has not an integer part in  $M_i$ . Then by combining the Integer Part Lemma of Boughattas [2] with the  $\widehat{\mathbb{Z}}$ -construction, we obtain  $M_{i+1}$  with its parsimonious homomorphism extension to  $\widehat{\mathbb{Z}}$ , such that  $b_{\alpha_i}$  has an integer part in  $M_{i+1}$ . Also suppose  $M_j$  has been constructed and at stage  $j + 1$  we want to do the F-construction. We seek the least ordinal  $\alpha_j$ , such that  $b_{\alpha_j}$  is larger than any element of  $M_j$ . Then the F-construction can be done at this step. At limit stages we take union. Moreover, Lemma 9.1, Theorem 9.4 and Theorem 9.8 of (Smith [8]) will guarantee preserving parsimony of homomorphisms and factoriality at limit stages of length  $\leq \omega_1$ . Now there is no obstacle for  $M = \bigcup_{i < \omega_1} M_i$  to be a Bezout integer part of  $K$ .  $\dashv$

THEOREM C.  $Bez + (OI)_p \not\vdash (OI)_{p+1}$ , when  $p \neq 3$ .

PROOF OF THEOREM C. In [4], we showed that if  $p \neq 3$ , there is an irreducible polynomial  $P(t)$  of degree  $p + 1$  over  $\mathbb{Q}$  such that  $P(t)$  has no root in  $RC_p(\mathbb{Q})$ . For  $p \geq 4$ ,  $P(t)$  was a polynomial with Galois group  $A_{p+1}$ . It is well known

that we can take  $P(t)$  as a monic polynomial with integer coefficients such that  $P(0) < 0$ . Let  $T$  be the following theory in the language of ordered field with the additional constant symbol  $a$ :

$$T \equiv (RCF)_p + \{a > k; k \in \mathbb{N}\} + \forall y \neg(Q(y) \leq 0 < Q(y+1)),$$

where  $Q(y) = a^{p+1}P(y/a)$ . We show that the field of Puiseux power series  $RC_p(\mathbb{Q})((x^{1/\mathbb{N}}))$  is a model of  $T$ , when interpreting  $a$  by  $x$ . Clearly by Theorem 2.4,  $RC_p(\mathbb{Q})((x^{1/\mathbb{N}}))$  is a  $p$ -real closed field. Also on the contrary suppose that there exists  $y \in RC_p(\mathbb{Q})((x^{1/\mathbb{N}}))$  such that  $(Q(y) \leq 0 < Q(y+1))$ . Therefore  $P(y/x) \leq 0 < P((y+1)/x)$ . It is easily seen that  $\deg_x(y/x)$  must be zero. So let  $y/x = \lambda + \sum_{-\infty < i < 0} c_i x^{i/q}$  in  $RC_p(\mathbb{Q})((x^{1/\mathbb{N}}))$ . This leads to  $P(\lambda) = 0$ , but  $\lambda \in RC_p(\mathbb{Q})$ , which is in contradiction to the choice of  $P(t)$ . So  $RC_p(\mathbb{Q})((x^{1/\mathbb{N}})) \models T$ .

Now that  $T$  is consistent, let  $K$  be an  $\omega_1$ -saturated model of  $T$ . By Lemma 3.14  $K$  has a Bezout integer part. Call it  $M$ . Since  $K \models (RCF)_p$ , then  $M \models (OI)_p$ . On the other hand there is  $n \in \mathbb{N}$  such that  $M \models Q(0) < 0 < Q(n[a]_M)$ , where  $[a]_M$  is the integer part of  $a$  in  $M$ . But  $K \models \forall y \neg(Q(y) \leq 0 < Q(y+1))$ , then  $M \models \forall y \neg(Q(y) \leq 0 < Q(y+1))$ , so by (Boughattas [1], Proposition A.I),  $M \models \neg(OI)_{p+1}$ . This ends the proof of  $Bez + (OI)_p \not\models (OI)_{p+1}$ .  $\dashv$

Proof of Theorem D goes the same way with the exception that we must replace Lemma 3.14 by the following Construction Lemma of Boughattas:

**THEOREM 3.15** (Boughattas [2]). *Suppose  $K$  is a saturated ordered field. Let  $\Lambda$  be an arbitrary subset of real algebraic elements in  $K$ . Then there exists  $X \subset K$  such that  $X$  is algebraic independent and  $\mathbb{Z}[\{rx; r \in \mathbb{Q}[\Lambda] \text{ and } x \in X\}]$  is an integer part of  $K$ .*

**THEOREM D.**  $(OI)_p + \neg(OI)_{p+1} + (N)_n + \neg(N)_{n+1}$  is consistent, when  $p \neq 3$ .

**PROOF OF THEOREM D.** We work with the same theory  $T$  and its saturated model  $K$  as in the proof of Theorem C. Choose  $\Lambda = \{\lambda\}$  and fix  $\lambda$  as in the proof of Theorem B, namely, if  $n \neq 3$ ,  $\lambda \in RC_{n+1}(\mathbb{Q})$ ,  $\lambda \notin RC_n(\mathbb{Q})$  and if  $n = 3$  choose  $\lambda$  as a root of an irreducible polynomial of degree 4 such that  $\lambda \notin RC_2(\mathbb{Q})$ . Then by Theorem 3.15, there exists  $X \subset K$  such that  $K$  has the integer part  $M = \mathbb{Z}[\{rx; r \in \mathbb{Q}(\lambda) \text{ and } x \in X\}]$ . To show  $M \models (N)_n + \neg(N)_{n+1}$ , we can repeat the proof of Theorem B, just replace  $x$  by  $X$  and replace  $A_{\langle S \rangle}$  by  $\mathbb{Q}(\lambda)$ .  $\mathbb{Q}(\lambda)[X]$  remains factorial and normal, so the proof works. By the last paragraph of the proof of the Theorem C, it is obvious that  $M \models (OI)_p + \neg(OI)_{p+1}$ .  $\dashv$

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# DIOPHANTINE CORRECT OPEN INDUCTION

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**Abstract.** We give an induction-free axiom system for diophantine correct open induction. We reduce the problem of whether a finitely generated ring of Puiseux polynomials is diophantine correct to a problem about the value-distribution of a tuple of semialgebraic functions with integer arguments. We use this result, and a theorem of Bergelson and Leibman on generalized polynomials, to identify a class of diophantine correct subrings of the field of descending Puiseux series with real coefficients.

## Introduction.

**Background.** A model of open induction is a discretely ordered ring whose semiring of non-negative elements satisfies the induction axioms for open<sup>1</sup> formulas.

Equivalently, a model of open induction is a discretely ordered ring  $R$ , with real closure  $F$ , such that every element of  $F$  lies at a finite distance from some element of  $R$ .<sup>2</sup>

The surprising equivalence between these two notions was discovered by Shepherdson [7]. This equivalence enabled him to identify naturally occurring models of open induction made from Puiseux polynomials. Let  $F$  be the field of descending<sup>3</sup> Puiseux series with coefficients in some fixed real closed subfield of  $\mathbb{R}$ . Puiseux's theorem implies that  $F$  is real closed. There is a unique ordering on  $F$ , in which the positive elements are the series with positive leading coefficients. Define an “integer part” function on  $F$  as follows:

$$\left\lfloor \sum_{i < M} a_i t^{i/D} \right\rfloor = \lfloor a_0 \rfloor + \sum_{i > 0} a_i t^{i/D}$$

where  $\lfloor a_0 \rfloor$  is the usual integer part of the real number  $a_0$ .

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<sup>1</sup>A formula is “open” if it is quantifier-free.

<sup>2</sup>Consequently, the inequality  $r \leq x < r + 1$  defines a function  $r = \lfloor x \rfloor$  from  $F$  onto  $R$ . This function is the natural counterpart of the usual integer part operator from  $\mathbb{R}$  onto  $\mathbb{Z}$ .

<sup>3</sup>A descending Puiseux series with real coefficients has the form  $\sum_{i < M} a_i t^{i/D}$ , where  $M$  is an integer,  $D$  is a positive integer, and the  $a_i$  are real.

The image of  $\lfloor \cdot \rfloor$  is the subring  $R$  of  $F$  consisting of all Puiseux polynomials with constant terms in  $\mathbb{Z}$ . Since every Puiseux series is a finite distance from its leading Puiseux polynomial, it is immediate that every element of  $F$  is a finite distance from some element of  $R$ . The discreteness of the ordering on  $R$  is a consequence of the polynomials in  $R$  having integer constant terms. By Shepherdson's equivalence,  $R$  is a model of open induction.

There has been some effort to find other models of open induction in the field of real Puiseux series  $F$ , satisfying additional properties of the ordered ring of integers. Perhaps the most extreme possibility in this regard is that  $F$  contains a model of open induction that is diophantine correct. We shall say that an ordered ring is *diophantine correct* if it satisfies every universal sentence true in the ordered ring of integers. We refer to the theory of diophantine correct models of open induction as *DOI*. To make this notion precise, we shall assume that ordered rings have signature  $(+ - \cdot \leq 0 1)$ . All formulas will be assumed to be of this type. Diophantine correctness amounts to the requirement that an ordered ring not satisfy any system of polynomial equations and inequalities that has no solution in the ring of integers.

Shepherdson's models are not diophantine correct.<sup>4</sup> However, there are other models of open induction in the field of real Puiseux series, notably the rings constructed by Berarducci and Otero [1], which are not obviously not diophantine correct. More generally, it seems to be unknown whether the field of real Puiseux series has a diophantine correct integer part.

**PROBLEM.** *Let  $F$  be the field of Puiseux series with coefficients in a real closed subfield  $E$  of  $\mathbb{R}$  of positive transcendence degree over the rationals. Must (Can)  $F$  contain a model of DOI other than  $\mathbb{Z}$ ?*

We prove in Section 2 that the field  $E$  must have positive transcendence degree, otherwise the only model of DOI contained in  $F$  is  $\mathbb{Z}$ .

**Wilkie's theorems and the models of Berarducci and Otero.** Wilkie [10] gave necessary and sufficient conditions for an ordinary (unordered) ring  $R$  to have an expansion to an ordered ring that extends to a model of open induction. These conditions are

- (1) For each prime  $p$ , there must be a homomorphism  $h_p : R \rightarrow \mathbb{Z}_p$ , where  $\mathbb{Z}_p$  is the ring of  $p$ -adic integers.<sup>5</sup>
- (2) It must be possible to discretely order the ring  $R$ .

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<sup>4</sup>For example, there are positive solutions of the equation  $x^2 = 2y^2$  via the Puiseux polynomials  $x = \sqrt{2}t$  and  $y = t$ .

<sup>5</sup>This is equivalent to the condition that for every positive integer  $n$  and every prime  $p$  there is a homomorphism from  $R$  onto the ring  $\mathbb{Z}/p^n\mathbb{Z}$ .

These conditions are independent. For example, the ring  $R = \mathbb{Z}[t, (t^2 + 1)/3]$  is discretely ordered by making  $t$  infinite.<sup>6</sup> But the equation  $1 + x^2 = 3y$  is solvable in  $R$  but not in  $\mathbb{Z}_3$ , so there is no homomorphism from  $R$  into  $\mathbb{Z}_3$ .

Conversely, let  $g(t)$  be the polynomial  $(t^2 - 13)(t^2 - 17)(t^2 - 221)$ . The ring  $R = \mathbb{Z}[t, t + 1/(1 + g(t)^2)]$  can be mapped homomorphically to  $\mathbb{Z}_p$  for every  $p$ ,<sup>7</sup> but cannot be discretely ordered: The second generator minus the first is between two integers if  $t$  is not.

Wilkie [10] gave conditions under which an ordered ring can be extended so as to preserve these two conditions (using a single ordering.) We paraphrase his results.

**THEOREM (Wilkie's Extension Theorem).** *Let  $R$  be discretely ordered ring. Suppose that for every prime  $p$  there is a homomorphism  $h_p : R \rightarrow \mathbb{Z}_p$ . Let  $F$  be a real closed field containing  $R$  and let  $s \in F$ . Then*

- (1) *If  $s$  is not a finite distance from any element of  $R[\mathbb{Q}]$ , and  $s$  is not infinitely close to any element of the real closure of  $R$  in  $F$ , then  $R[s]$  is discretely ordered as a subring of  $F$ , and the homomorphisms  $h_p$  can be extended to  $R[s]$  by assigning  $p$ -adic values to  $s$  arbitrarily.*
- (2) *If  $s \in R[\mathbb{Q}]$ , then choose  $n \in \mathbb{Z}$  so that  $ns \in R$ . Choose  $m \in \mathbb{Z}$  so that  $n$  divides  $h_p(ns) - m$  in  $\mathbb{Z}_p$ , for every prime  $p$ . Put  $r = (ns - m)/n$ . Then  $R[r]$  is discretely ordered, and the homomorphisms  $h_p$  extend to  $R[r]$  via  $h_p(r) = (h_p(ns) - m)/n$ .*

The choice of  $m$  in Case (2) is always possible because  $n$  will be a unit in  $\mathbb{Z}_p$  for all  $p$  prime to  $n$ . Suppose  $n$  has prime decomposition  $\prod p_i^{e_i}$ . For each of the prime divisors  $p_i$  of  $n$ , choose  $m_i \in \mathbb{Z}$  so close<sup>8</sup> to  $h_{p_i}(ns)$  that  $m_i \equiv h_{p_i}(ns) \pmod{p_i^{e_i}}$ . Then use the Chinese remainder theorem to get  $m \equiv m_i \pmod{p_i^{e_i}}$ .

The point is that starting with an ordered ring  $R$  and homomorphisms  $h_p$  as above, one can extend  $R$  to a model of open induction by repeatedly adjoining missing integer parts of elements of a real closure of  $R$ . We give an example of how this is done. Let  $R = \mathbb{Z}[t]$ , and let  $F$  be the field of real Puiseux series. Let  $h_p : R \rightarrow \mathbb{Z}_p$  be the homomorphism given by the rule<sup>9</sup>

$$h_p(f(t)) = f(1 + p + p^2 + \dots).$$

Think of the polynomial  $s = t^2/36$  as an element of some fixed real closure of  $R$ . Then  $s$  has no integer part in  $R$ . We shall adjoin an integer part *via* Case (2).

<sup>6</sup>To prove discreteness, first show that  $R/3R$  is a nine-element field. If  $H$  is a polynomial with integer coefficients and if  $r = H(t, (t^2 + 1)/3)$  is finite but not an integer, then  $r$  has the form  $a/3^n$ , where  $3 \nmid a$  and  $n > 0$ . Map the equation  $3^n H(t, (t^2 + 1)/3) = a$  to  $R/3R$  to get a contradiction.

<sup>7</sup>To find a homomorphism  $h$  from  $R$  into  $\mathbb{Z}_p$ , use the fact that the polynomial  $g(x)$  has  $p$ -adic zeros for all  $p$ . See [3]. Set  $h_p(x) = r$ , where  $r$  is a  $p$ -adic zero of  $g$ , and set  $h_p(x + 1/(1 + g^2)) = r + 1$ , and show that  $h_p$  extends to a homomorphism from  $R$  into  $\mathbb{Z}_p$ .

<sup>8</sup>In the sense of the  $p$ -adic metric.

<sup>9</sup> $h_p$  is the unique homomorphism from  $R$  into  $\mathbb{Z}_p$  taking  $t$  to  $1/(1 - p) = 1 + p + p^2 + \dots$ .

Since  $36s \in R$ , we must find  $m \in \mathbb{Z}$  so close to  $h_p(36s) = 1 + 2p + 3p^2 + \dots$  that 36 will divide  $h_p(36s) - m$ . This is only an issue for  $p = 2, 3$ , since otherwise 36 is a unit. It is enough to solve the congruences

$$\begin{aligned} m &\equiv 1 + 2 \cdot 2^1 \pmod{2^2} \\ m &\equiv 1 + 2 \cdot 3^1 \pmod{3^2}. \end{aligned}$$

Here  $m = 25$  does the job. Thus we adjoin  $(36s - 25)/36 = (t^2 - 25)/36$  to  $R$ .

To continue, the element  $\sqrt{2}t$  is not within a finite distance of any element of the ring  $R_1 = \mathbb{Z}[t, (t^2 - 25)/36]$ . We can fix that *via* Case (1) by adjoining  $\sqrt{2}t + r$ , where  $r$  is any transcendental real number. The fact that  $r$  is transcendental insures that  $\sqrt{2}t + r$  is not infinitely close to any element of the real closure of  $R_1$ . We can extend the maps  $h_p$  to  $R_1$  by assigning  $p$ -adic values to  $\sqrt{2}t + r$  arbitrarily.

The models of open induction in [1] are constructed, with some careful bookkeeping, by iterating the procedure just described. Up to isomorphism, the result is a polynomial ring  $R$  over  $\mathbb{Z}$  in infinitely many variables that becomes a model of open induction by adjoining elements  $r/n$  ( $r \in R, n \in \mathbb{Z}$ ) in accordance with Case (2) of Wilkie's extension theorem. We suspect that all of these rings are diophantine correct. As we shall see, the question turns on how subtle are the polynomial identities that can hold on the integer points of a certain class of semialgebraic sets.

The plan of the paper is as follows. In Section 1 we give a simplified axiom system for *DOI*. In Section 2 we give number-theoretic conditions for a finitely generated ring of Puiseux polynomials to be diophantine correct: We show how the diophantine correctness of such a ring is a problem about the distributions of the values at integer points of certain tuples of generalized polynomials.<sup>10</sup> In Section 3 we give some recent results on generalized polynomials, and in Section 4 we use these results to give a class of ordered rings of Puiseux polynomials for which consistency with the axioms of open induction and diophantine correctness are equivalent.

**§1. Axioms for *DOI*.** In this section we prove that *DOI* is equivalent to all true (in  $\mathbb{Z}$ ) sentences  $\forall \bar{x} \exists y \phi$ , with  $\phi$  an open formula. The underlying reason for this fact is that compositions of the integer part operator with semialgebraic functions suffice to witness the existential quantifier in every true  $\forall \bar{x} \exists y$  sentence.

**THEOREM 1.1.** *DOI is axiomatized by the set of all sentences true in the ordered ring of integers of the form  $\forall x_1 \forall x_2 \dots \forall x_n \exists y \phi$ , with  $\phi$  an open formula.*

<sup>10</sup>A generalized polynomial is an expression made from arbitrary compositions of real polynomials with the integer part operator. See [2].

The proof requires two lemmas. The first is a parametric version of the fact that definable subsets in real closed fields are finite unions of intervals.

Let  $F$  be a real closed field and  $\phi(x, \bar{y})$  a formula. For each  $\bar{r} \in F$ , the subset of  $F$  defined by  $\phi(x, \bar{r})$  can be expressed as a finite union  $I_{1, \bar{r}} \cup \dots \cup I_{n, \bar{r}}$ , where the  $I_{i, \bar{r}}$  are either singletons or open intervals with endpoints in  $F \cup \{\pm\infty\}$ . We shall require the fact that for each  $\phi$  there are formulas  $\gamma_i(x, \bar{y})$  such that for every  $\bar{r}$ , the  $\gamma_i(x, \bar{r})$  define such intervals  $I_{i, \bar{r}}$ .

LEMMA 1.2. *Let  $\phi(x, \bar{y})$  be a formula in the language of ordered rings. Then there is a finite list of open formulas  $\gamma_i(x, \bar{y})$  such that the theory of real closed fields proves the following sentences:*

- (1)  $\forall x, \bar{y} (\phi(x, \bar{y}) \leftrightarrow \bigvee_i \gamma_i(x, \bar{y}))$
- (2)  $\bigwedge_i \forall \bar{y} ((\neg \exists x \gamma_i(x, \bar{y})) \vee (\exists! x \gamma_i(x, \bar{y})) \vee (\exists z \forall x (\gamma_i(x, \bar{y}) \leftrightarrow x < z)) \vee (\exists z \forall x (\gamma_i(x, \bar{y}) \leftrightarrow x > z)) \vee (\exists z, w \forall x (\gamma_i(x, \bar{y}) \leftrightarrow z < x < w)))$

Formula (1) asserts that for any tuple  $\bar{r}$  in a real closed field, the set defined by  $\phi(x, \bar{r})$  is the union of the sets defined by the  $\gamma_i(x, \bar{r})$ . Formula (2) asserts that each set defined by  $\gamma_i(x, \bar{r})$  is either empty, or a singleton, or an open interval.

PROOF. This is a well-known consequence of Thom's Lemma. See [8].  $\dashv$

The next Lemma shows that in models of *OI*, a one-quantifier universal formula is equivalent to an existential formula.

LEMMA 1.3. *For every formula  $\forall x \phi(x, \bar{y})$  with  $\phi$  open, there are open formulas  $\psi_i(x_i, \bar{y})$  such that*

$$OI \vdash \forall \bar{y} \left( (\forall x \phi(x, \bar{y})) \leftrightarrow \bigwedge_i \exists x_i \psi_i(x_i, \bar{y}) \right).$$

The idea of the proof is as follows: If the formula  $\forall x \phi(x, \bar{r})$  holds in some model  $R$  of open induction, with  $\bar{r} \in R$ , then the formula  $\phi(x, \bar{r})$  must hold for all elements  $x$  of the real closure of  $R$ , except for finitely many intervals  $U_i$  of length at most 1. The existential formula  $\exists x_i \psi_i(x_i, \bar{y})$  says that for some  $e_i \in R$ , the set  $U_i$  is included in the open interval  $(e_i, e_i + 1)$ .

PROOF OF LEMMA 1.3. Let  $\gamma_i$  be the formulas given by the statement of Lemma 1.2, using  $\neg\phi$  in place of  $\phi$ . Thus Formula (1) of Lemma 1.2 now reads

$$\forall x, \bar{y} \left( \neg\phi(x, \bar{y}) \leftrightarrow \bigvee_i \gamma_i(x, \bar{y}) \right). \quad (*)$$

By Tarski's Theorem, choose quantifier free formulas  $\alpha_i(z, \bar{y})$  and  $\beta_i(z, \bar{y})$  such that the theory of real closed fields proves

$$\forall z, \bar{y} (\alpha_i(z, \bar{y}) \leftrightarrow \forall w (\gamma_i(w, \bar{y}) \rightarrow z < w))$$

and

$$\forall z, \bar{y} (\beta_i(z, \bar{y}) \leftrightarrow \forall w (\gamma_i(w, \bar{y}) \rightarrow w < z)).$$

If  $F$  is a real closed field, and if  $\bar{r} \in F$ , then  $\alpha_i(x_i, \bar{r})$  defines all elements  $x_i$  of  $F$  such that  $x_i$  is less than any element of the set defined by  $\gamma_i(x, \bar{r})$ . Similarly,  $\beta_i(x_i, \bar{r})$  defines all elements  $x_i$  of  $F$  such that  $x_i$  is greater than any element of the set defined by  $\gamma_i(x, \bar{r})$ .

Define the formula  $\psi_i(x_i, \bar{y})$  required by the conclusion of the Lemma to be

$$\alpha_i(x_i, \bar{y}) \wedge \beta_i(x_i + 1, \bar{y}).$$

We must prove that the equivalence

$$\forall \bar{y} ((\forall x \phi(x, \bar{y})) \leftrightarrow \bigwedge_i \exists x_i \psi_i(x_i, \bar{y}))$$

holds in every model of open induction  $R$ .

For the left-to-right direction, let  $\bar{r}$  be a tuple from  $R$ , and suppose that  $R$  satisfies  $\forall x \phi(x, \bar{r})$ . For each  $i$  we must find  $g$  in  $R$  such that

$$R \models \alpha_i(g, \bar{r}) \wedge \beta_i(g + 1, \bar{r}). \quad (**)$$

Let  $F$  be a real closure of  $R$ . Let  $I_{i, \bar{r}}$  be the open interval of  $F$  defined by the formula  $\gamma_i(x, \bar{r})$ .

The interval  $I_{i, \bar{r}}$  cannot be unbounded: It must have both endpoints in  $F$ . Otherwise  $I_{i, \bar{r}}$  would meet  $R$ .<sup>11</sup> If  $I_{i, \bar{r}}$  did meet  $R$ , then the universal sentence  $(*)$ , would give an element  $e \in R$  for which  $\neg \phi(e, \bar{r})$  holds, contrary to our assumption that  $R \models \forall x \phi(x, \bar{r})$ . Therefore  $I_{i, \bar{r}}$  is a bounded open interval.

If the interval  $I_{i, \bar{r}}$  is empty, then every  $g \in R$  will trivially satisfy condition  $(**)$ , and the proof will be complete. Therefore, we can assume that  $I_{i, \bar{r}}$  is nonempty. Formula  $(*)$  then implies that the half-open intervals defined by the formulas  $\alpha_i(x_i, \bar{r})$  and  $\beta_i(x_i + 1, \bar{r})$  will each have an endpoint in  $F$ , i.e., they will not be of the form  $(-\infty, \infty)$ .

The least number principle for open induction<sup>12</sup> implies that there is a greatest element  $g \in R$  such that  $R \models \alpha_i(g, \bar{r})$ . The maximality of  $g$  implies that  $R \models \neg \alpha_i(g + 1, \bar{r})$ . Hence  $g + 1$  is at least as large as some element of  $I_{i, \bar{r}}$ . Since  $I_{i, \bar{r}}$  is disjoint from  $R$ , it follows that  $g + 1$  is greater than every element of  $I_{i, \bar{r}}$ . Therefore  $\beta_i(g + 1, \bar{r})$  holds in  $R$ . We have found  $g$  satisfying the required condition  $(**)$ .

<sup>11</sup>If  $R$  is an ordered ring and  $F$  is a real closure of  $R$ , then  $R$  is cofinal in  $F$ . [4].

<sup>12</sup>In a model of open induction  $R$ , if a non-empty set  $S \subseteq R$  is defined, possibly with parameters, by an open formula, and if  $S$  is bounded below, say by  $b$ , then  $S$  has a least element. The reason is that otherwise if  $s \in S$  then the set of non-negative  $x \in R$  such that  $x + b \leq s$  is inductive.

For the right-to-left direction of the equivalence, assume that for every  $i$ , we have elements  $b_i \in R$  such that

$$R \models \alpha_i(b_i, \bar{a}) \wedge \beta_i(b_i + 1, \bar{a}).$$

This same formula will hold in  $F$ , hence for each  $i$ ,

$$F \models \forall w (\gamma_i(w, \bar{a}) \rightarrow b_i < w) \wedge \forall w (\gamma_i(w, \bar{a}) \rightarrow w < b_i + 1).$$

The last displayed statement asserts that every element  $b$  of  $F$  satisfying  $\gamma_i(b, \bar{a})$  lies between  $b_i$  and  $b_i + 1$ . But no element of  $R$  lies between  $b_i$  and  $b_i + 1$ . Therefore for every  $b \in R$ ,

$$R \models \neg \bigvee_i \gamma_i(b, \bar{a}).$$

This assertion, together with  $(*)$ , gives the conclusion  $R \models \forall x \phi(x, \bar{a})$ .  $\dashv$

**PROOF OF THEOREM 1.1.** Let  $T$  be the theory of all sentences true in  $\mathbb{Z}$  of the form  $\forall x_1 \forall x_2 \dots \forall x_n \exists y \phi$ , with  $\phi$  an open formula. We prove the equivalence  $T \Leftrightarrow DOI$ .

$T \Rightarrow DOI$  :

It is immediate that  $T \Rightarrow DOR + \forall_1(\mathbb{Z})$ . It remains to verify that  $T$  proves all instances of the induction scheme for open formulas. For each open formula  $\phi$ , the induction axiom

$$\forall \bar{x} ((\phi(\bar{x}, 0) \wedge \forall y \geq 0 (\phi(\bar{x}, y) \rightarrow \phi(\bar{x}, y + 1))) \rightarrow \forall z \geq 0 \phi(\bar{x}, z))$$

is logically equivalent to

$$\forall \bar{x} \forall z \exists y (z \geq 0 \rightarrow (y \geq 0 \wedge \phi(\bar{x}, 0) \wedge ((\phi(\bar{x}, y) \rightarrow \phi(\bar{x}, y + 1)) \rightarrow \phi(\bar{x}, z)))).$$

The latter belongs to  $T$ .

$DOI \Rightarrow T$  :

Suppose that  $R \models DOI$ . Let  $\phi(\bar{x}, y)$  be an open formula such that

$$\mathbb{Z} \models \forall \bar{x} \exists y \phi(\bar{x}, y).$$

We prove that  $R \models \forall \bar{x} \exists y \phi(\bar{x}, y)$ .

By Lemma 1.3, there are open formulas  $\psi_i$  such that  $OI$  proves the equivalence

$$\forall \bar{x} \left( (\exists y \phi(\bar{x}, y)) \longleftrightarrow \bigvee_i \forall z_i \psi(\bar{x}, z_i) \right).$$

The last two displayed assertions imply that  $\mathbb{Z} \models \forall \bar{x} (\bigvee_i \forall z_i \psi(\bar{x}, z_i))$ . But  $R$  is diophantine correct, therefore  $R \models \forall \bar{x} (\bigvee_i \forall z_i \psi(\bar{x}, z_i))$ . Since  $R$  is a model of  $OI$ , the above equivalence holds in  $R$ . Therefore  $R \models \forall \bar{x} \exists y \phi(\bar{x}, y)$ .  $\dashv$

**§2. Diophantine correct rings of Puiseux polynomials.** Let  $\mathcal{P}$  denote the ring of Puiseux polynomials with real coefficients. We will think of Puiseux polynomials interchangeably as formal objects and as functions from the positive reals to the reals. The following theorem describes the conditions for a finitely generated subring of  $\mathcal{P}$  to be diophantine correct, in terms of the coefficients of a list of generating polynomials. We shall use this theorem to investigate the diophantine correct subrings of  $\mathcal{P}$ . To simplify notation we temporarily assume that not all the coefficients of the generating polynomials are algebraic numbers.

**THEOREM 2.1.** *Let  $f_1 \dots f_n \in \mathcal{P}$ . Assume that the  $f_i$  are non-constant, and that the field  $F$  generated by the coefficients of the  $f_i$  has transcendence degree at least 1 over  $\mathbb{Q}$ . Let  $\bar{r} = r_1 \dots r_l$  be a transcendence basis for  $F$  over  $\mathbb{Q}$ . Then*

- (1) *There is an open formula  $\theta(x_1 \dots x_l, y_1 \dots y_n)$  such that  $\theta(\bar{r}, \bar{y})$  holds in  $\mathbb{R}$  at  $\bar{y}$  if and only if for some real  $t \geq 1$ ,  $\bigwedge_i y_i = f_i(t)$ .*
- (2) *Choose  $\theta$  as in (1). The ring  $\mathbb{Z}[\bar{f}]$  is diophantine correct if and only if for every open neighborhood  $U \subseteq \mathbb{R}^l$  of  $\bar{r}$  and for every positive integer  $M$ , there are points  $\bar{s} \in U$  and integers  $\bar{m}$  such that  $\min_i |m_i| > M$  and  $\mathbb{R} \models \theta(\bar{s}, \bar{m})$ .*

We give two examples to show how Theorem 2.1 can be used to determine whether a given subring of  $\mathcal{P}$  is diophantine correct.

**EXAMPLE 2.2.** Let  $R = \mathbb{Z}[t, f(t) - r_1]$ , where  $r_1$  is a real transcendental and  $f$  is a polynomial with algebraic coefficients. The formula  $\theta(r_1, y_1, y_2)$  expresses the condition

$$\exists t \geq 1 (y_1 = t \wedge y_2 = f(y_1) - r_1).$$

Eliminating the quantifier we obtain<sup>13</sup>

$$\theta(r_1, y_1, y_2) : y_1 \geq 1 \wedge f(y_1) - y_2 = r_1.$$

It follows that the ring  $R$  is diophantine correct if and only if there are positive integers  $\bar{y}$  making  $f(y_1) - y_2$  arbitrarily close to  $r_1$ .

It is known<sup>14</sup> that the values of  $f(y_1) - y_2$  are either dense in the real line, if  $f$  has an irrational coefficient other than its constant term, or otherwise discrete. In the former case  $R$  is diophantine correct. In the latter case  $f(y_1) - y_2$  could only approach  $r_1$  by being equal to  $r_1$ , which is impossible since  $r_1$  is transcendental.

**EXAMPLE 2.3.** Let  $R = \mathbb{Z}[t, \sqrt{2}t - r, 2\sqrt{2}rt - s]$ , with  $r$  and  $s$  algebraically independent. Then  $R$  is diophantine correct if and only if the point

$$(\sqrt{2}y_1 - y_2, 2\sqrt{2}y_1(\sqrt{2} - y_2)y_1 - y_3) \tag{*}$$

<sup>13</sup>For the sake of clarity we neglect the translation into the language of ordered rings.

<sup>14</sup>This is a consequence of Weyl's Theorem on uniform distribution. See [5], p. 71.

can be made arbitrarily close to  $(r, s)$ . This is a non-linear approximation problem, and there is no well-developed theory of such problems. In this case the identity

$$(\sqrt{2}y_1 - y_2)^2 = (2\sqrt{2}y_1(\sqrt{2} - y_2)y_1 - y_3) - (2x^2 - y^2 - y_3)$$

implies that the point  $(*)$  cannot tend to the pair  $(r, s)$  unless  $r^2 - s$  is an integer. Hence the requirement that  $r$  and  $s$  be algebraically independent cannot be met.

The most general case of Theorem 2.1 cannot be written down explicitly, because the algebraic relations between coefficients can be arbitrarily complex. But, following the notation of Theorem 2.1, the fact that the  $r_i$  are algebraically independent implies that in the relation  $\theta(\bar{x}, \bar{y})$ , if  $\bar{x}$  is restricted to a small enough neighborhood of  $\bar{r}$  then each  $x_i$  is a semialgebraic function of  $\bar{y}$ .<sup>15</sup> Therefore the problem of whether a finitely generated ring of Puiseux polynomials is diophantine correct always has the form: “Are there tuples of integers  $\bar{y}$  such that the points  $(\sigma_1(\bar{y}), \dots, \sigma_n(\bar{y}))$  tend to the point  $\bar{r}$ ?” where the  $\sigma_i$  are semialgebraic functions.

This general type of problem is undecidable, since it contains Hilbert’s tenth problem.<sup>16</sup> But the rings that we actually want to use to construct models of open induction have a special form, which leads to a restricted class of problems that may well be decidable. (See Section 3.)

We return to Theorem 2.1, and the conditions for  $\mathbb{Z}[\bar{f}]$  to be diophantine correct. The idea of the proof is to think of the polynomials  $f_i(t)$  as functions of both  $t$  and  $\bar{r}$ . If  $\phi$  is an open formula, then the statement that  $\phi(\bar{f})$  holds in  $\mathbb{Z}[\bar{f}]$  can be expressed as another open formula  $\psi(\bar{r})$ . The latter must hold on an entire neighborhood of  $\bar{r}$ , since the  $r_i$  are algebraically independent. If  $\mathbb{Z}[\bar{f}] \models \phi(\bar{f})$  then we can try to perturb the  $r_i$  a tiny bit for very large  $t$  so as to make the values  $f_i(\bar{r}, t)$  into integers. The formula  $\theta$  expresses the relation between the perturbed values of  $\bar{r}$  and the resulting integer values of  $\bar{f}$ .

We hope that this explanation motivates the use of following three Lemmas. We omit the straightforward proofs.

**LEMMA 2.4.** *Let  $f_1, f_2 \dots f_n \in \mathcal{P}$ . Let  $\phi(\bar{x})$  be an open formula. Then  $\mathbb{Z}[\bar{f}] \models \phi(\bar{f})$  if and only if for all sufficiently positive  $t \in \mathbb{R}$ , the formula  $\phi(\bar{x})$  holds in  $\mathbb{R}$  at the tuple of real numbers  $\bar{f}(t)$ .*  $\dashv$

**LEMMA 2.5.** *Let  $\bar{f} = f_1(t) \dots f_n(t) \in \mathcal{P}$ . The ordered ring  $\mathbb{Z}[\bar{f}]$  is diophantine correct if and only if for every open formula  $\phi(\bar{x})$  such that  $\mathbb{Z}[\bar{f}] \models \phi(\bar{f})$ , there exists  $\bar{m} \in \mathbb{Z}$  such that  $\mathbb{Z} \models \phi(\bar{m})$ .*  $\dashv$

<sup>15</sup>See [8], p. 32, Lemma 1.3.

<sup>16</sup>For example,  $f(y_1, \dots, y_{n-2})^2 + (\sqrt{2}y_{n-1} - y_n)^2$  can be made arbitrarily close to a given number  $r$  between 0 and 1 if and only if  $f$  has an integer zero.

LEMMA 2.6. *Suppose that  $\phi(\bar{x})$  is a formula and  $\bar{r} \in \mathbb{R}^n$  is a tuple of algebraically independent real numbers.<sup>17</sup> If  $\mathbb{R} \models \phi(\bar{r})$ , then there is a neighborhood  $U$  of  $\bar{r}$  such that for every  $\bar{u} \in U$ ,  $\mathbb{R} \models \phi(\bar{u})$ .  $\dashv$*

PROOF OF THEOREM 2.1. To prove Part (1), let  $f_i(t) = g_i(\bar{c}, t)$ , where  $g_i$  is a polynomial with integer coefficients, and the  $c_i$  are algebraic over the field  $\mathbb{Q}(\bar{r})$ . Then  $c_i$  can be defined from the  $r_i$ , say by a formula  $\gamma_i(\bar{r}, \bar{c})$ . Eliminate quantifiers from the formula

$$\exists t \geq 1 \exists \bar{w} (y_i = g_i(\bar{w}, t) \wedge \gamma_i(\bar{w}, \bar{x}))$$

to obtain an open formula  $\theta_i(\bar{x}, y_i)$ , and let  $\theta$  be the conjunction of the  $\theta_i$ .

To prove the left-to-right direction of Part (2), assume that  $\mathbb{Z}[\bar{f}]$  is diophantine correct, and let  $\bar{r}$  and  $\theta$  be as in Part (1). Let  $U \subseteq \mathbb{R}^l$  be an open set containing  $\bar{r}$ , and fix a positive integer  $M$ . We must find  $\bar{s} \in U$  and  $\bar{m} \in \mathbb{Z}^n$ , with  $|m_i| > M$ , such that  $\theta(\bar{s}, \bar{m})$  holds in  $\mathbb{R}$ .

Since  $U$  is open, there is a formula  $\gamma(\bar{x})$  which holds at  $\bar{r}$ , and which defines an open set included in  $U$ . By Tarski's theorem, there is an open formula  $\theta_1(\bar{y})$  such that

$$RCF \vdash \theta_1(\bar{y}) \leftrightarrow \exists \bar{x} \left( \left( \bigwedge_i |y_i| > M \right) \wedge \gamma(\bar{x}) \wedge \theta(\bar{x}, \bar{y}) \right).$$

The formula  $\theta_1(f_1(t), \dots, f_n(t))$  must hold in  $\mathbb{R}$  for all sufficiently large  $t$ , since the functions  $|f_i(t)|$  tend to infinity with  $t$ , and since moreover we can witness the above existential quantifier with  $\bar{r}$ . Therefore, by Lemma 2.4,  $\mathbb{Z}[\bar{f}] \models \theta_1(\bar{f})$ .

Since  $\mathbb{Z}[\bar{f}]$  is diophantine correct, it follows that there are integers  $\bar{m} \in \mathbb{Z}^n$  satisfying  $\theta_1(\bar{y})$ . Substituting  $\bar{m}$  for  $\bar{y}$  in the above equivalence, the right hand side gives a tuple  $\bar{s} \in \mathbb{R}$  such that

$$\mathbb{R} \models \left( \left( \bigwedge_i |m_i| > M \right) \wedge \gamma(\bar{s}) \wedge \theta(\bar{s}, \bar{m}) \right).$$

Since  $\gamma(\bar{x})$  defines a subset of  $U$ , the displayed statement confirms that  $\bar{s}$  and  $\bar{m}$  are the tuples required.

To prove the right-to-left direction of Part (2), let  $\phi$  be an open formula such that  $\mathbb{R}[\bar{f}] \models \phi(\bar{f})$ . We prove that there are integers  $\bar{m}$  such that  $\phi(\bar{m})$  holds in  $\mathbb{Z}$ . It will follow immediately from Lemma 2.5 that  $\mathbb{Z}[\bar{f}]$  is diophantine correct.

Since  $\phi$  is open and since  $\phi(\bar{f})$  holds in  $\mathbb{Z}[\bar{f}]$ , it follows from Lemma 2.4 that  $\phi(f_1(t), \dots, f_n(t))$  holds in  $\mathbb{R}$  for all sufficiently positive  $t$ . Choose  $k > 1$  such that  $\phi(f_1(t), \dots, f_n(t))$  holds in  $\mathbb{R}$  for  $t > k$ .

<sup>17</sup>Algebraically independent over  $\mathbb{Q}$ .

The set of points  $(f_1(t), \dots, f_n(t))$  with  $1 \leq t \leq k$  is bounded. Therefore we can choose  $M \in \mathbb{Z}$  so large that if  $t \geq 1$  and if  $\min_i |f_i(t)| > M$ , then  $t > k$ . For this choice of  $M$ , the formula  $\psi(\bar{x})$  will hold in  $\mathbb{R}$  at  $\bar{r}$ , where  $\psi(\bar{x})$  is the formula

$$\forall \bar{y} \left( \left( \theta(\bar{x}, \bar{y}) \wedge \left( \bigwedge_i |y_i| > M \right) \right) \rightarrow \phi(\bar{y}) \right).$$

By Lemma 2.6, the subset of  $\mathbb{R}^l$  defined by  $\psi(\bar{x})$  must include a neighborhood  $U$  of  $\bar{r}$ . By hypothesis, we can choose  $\bar{s} \in U$  and  $\bar{m} \in \mathbb{Z}^n$  so that

$$\mathbb{R} \models \theta(\bar{s}, \bar{m}) \wedge \bigwedge_i |m_i| > M.$$

Instantiating the universal quantifier in  $\psi(\bar{s})$  with  $\bar{m}$ , we conclude that  $\phi(\bar{m})$  holds in  $\mathbb{Z}$ .  $\dashv$

**REMARK 2.7.** If the  $f_i$  have algebraic coefficients, then  $\mathbb{R}[\bar{f}]$  is diophantine correct if and only if there is a sequence of real numbers  $u_i$  tending to infinity such that  $\bar{f}(u_i) \in \mathbb{Z}^n$ . To prove this, one takes the transcendence basis  $\bar{r}$  to empty in the proof of Theorem 2.1 and one follows the proof, making all the necessary changes. This case is not important for our purposes because of the following fact.

**PROPOSITION 2.8.** *There are no models of DOI of transcendence degree one.*

**PROOF.** Suppose by way of contradiction that  $R$  is a model of DOI of transcendence degree 1. Let  $a$  be a non-standard element of  $R$ . Let  $b \in R$  be an integer part of  $\sqrt[3]{2}a$ . Then there is a nonzero polynomial  $p$  with integer coefficients such that  $p(a, b) = 0$ . We can assume that  $p$  is irreducible over the rationals. Since  $R$  is diophantine correct, the equation  $p(x, y) = 0$  must have infinitely many standard solutions. We shall prove that this is impossible.

Write  $p = p_0 + \dots + p_n$ , where  $p_i$  is homogeneous of degree  $i$ , and  $p_n \neq 0$ . Then  $p(a, b)$  has the form  $\sum_{i=0}^n p_i(1, b/a)a^i$ .

Observe that  $b/a$  is finite, in fact infinitely close to  $\sqrt[3]{2}$ , hence all the values  $p_i(1, b/a)$  are finite. It follows that for  $p(a, b)$  to be zero,  $p_n(1, b/a)$  must be infinitesimal; otherwise  $p_n(1, b/a)a^n$  would dominate all the other terms  $p_i(1, b/a)a^i$ , and then  $p(a, b)$  could not even be finite.

Since  $b/a$  is infinitely close to  $\sqrt[3]{2}$ , it follows that  $p_n(1, \sqrt[3]{2}) = 0$ . Since  $p_n$  has integer coefficients, the polynomial  $y^3 - 2$  must divide  $p_n(1, y)$ . It follows that  $y^3 - 2x^3$  divides  $p_n$ .

But if  $f(x, y)$  is any polynomial with integer coefficients irreducible over the rationals, and if  $f$  has infinitely many integer zeros, then the leading homogeneous part of  $f$  must be a constant multiple of a power of a linear or quadratic form.<sup>18</sup> This is not the case for  $p_n$ , thanks to the factor  $y^3 - 2x^3$ .

<sup>18</sup>See [6], p. 266.

Therefore  $p$  cannot have infinitely many integer solutions. This is the required contradiction.  $\dashv$

### §3. Generalized polynomials.

**Special sequences of polynomials.** We now focus on a restricted class of rings, which arise by adjoining sequences of integer parts using Wilkie's extension theorem (given in the Introduction.) A similar but more general type of sequence was defined in [1] to construct normal models of open induction.

**DEFINITION 3.1.** A sequence of polynomials is *special* if it has the form

$$f_0(t), f_1(t) - r_1, \dots, f_n(t) - r_n,$$

where

- (1)  $f_0(t) = t$ , and the coefficients of  $f_1(t)$  are algebraic.
- (2) The  $r_i$  are algebraically independent real numbers.
- (3) For  $i > 1$ , the polynomial  $f_i$  has the form  $g_i(t, r_1 \dots r_{i-1})$  where  $g_i$  is a polynomial with algebraic coefficients.

Note that a ring  $\mathbb{Z}[\bar{f}]$  generated by a special sequence contains the polynomial  $t$ . As a consequence, a polynomial is algebraic over  $\mathbb{Z}[\bar{f}]$  if and only if its coefficients are algebraic over the field generated by the coefficients of the  $f_i$ .

**EXAMPLE 3.2.** The sequence of polynomials  $t, \sqrt{2}t - r^2, rt - s$ , where  $r, s$  are algebraically independent real numbers, is not a special sequence because  $rt$  is not a polynomial in  $r^2$  and  $t$ . The sequence  $t, 2t - r, (r^2 + s)t - s$  is not a special sequence because  $(r^2 + s)t$  is not a polynomial in  $r$  and  $t$ .

The conditions for a ring generated by a special sequence to be diophantine correct can be written out explicitly.

**PROPOSITION 3.3.** Suppose that  $f_0(t), f_1(t) - r_1, \dots, f_n(t) - r_n$  is a special sequence, with  $0 < r_i < 1$ . Let  $R = \mathbb{Z}[\bar{f}]$ . Choose polynomials  $g_i(t, \bar{r})$  as in Item (3) of Definition 3.1.

Define the polynomials  $\sigma_i$  inductively as follows. Let  $\sigma_1(y_0) = f_1(y_0)$ . For  $i > 1$ , let

$$\sigma_i(y_0 \dots y_{i-1}) = g_i(y_0, \sigma_1(y_0) - y_1, \dots, \sigma_{i-1}(y_0 \dots y_{i-2}) - y_{i-1}).$$

Then

- (1)  $R$  is diophantine correct if and only if the system of inequalities

$$|\sigma_1(y_0) - y_1 - r_1| < \varepsilon$$

$$|\sigma_2(y_0, y_1) - y_2 - r_2| < \varepsilon$$

.....

$$|\sigma_n(y_0, y_1 \dots y_{n-1}) - y_n - r_n| < \varepsilon$$

has integer solutions  $y_i$  for every positive  $\varepsilon$ .

- (2) For all sufficiently small positive  $\varepsilon$ , if  $\bar{y}$  is a solution to the inequalities (1) then  $y_i = \lfloor \sigma_i(y_0 \dots y_{i-1}) \rfloor$ .

PROOF. Item (1) simply spells out Theorem 2.1 for rings generated by special sequences. Item (2) follows from the assumption that the  $r_i$  are in the interval  $(0, 1)$ , hence so are the values  $\sigma_i(y_0 \dots y_{i-1}) - y_i$  if  $\varepsilon$  is small enough.  $\dashv$

There is another way to think of the inequalities in Proposition 3.3. Since the equation  $y_i = \lfloor \sigma_i(y_0 \dots y_{i-1}) \rfloor$ , holds for all small enough  $\varepsilon$ , it follows that

$$\sigma_i(y_0 \dots y_{i-1}) - y_i = \{\sigma_i(y_0 \dots y_{i-1})\},$$

where  $\{\cdot\}$  is the fractional part operator, defined by  $\{x\} = x - \lfloor x \rfloor$ . Replacing  $y_1$  with  $\lfloor \sigma_1(y_0) \rfloor$  in the right hand side of the above equation and continuing in this fashion, we eventually obtain an expression for  $\{\sigma_i(y_0 \dots y_{i-1})\}$  as a function of  $y_0$  alone, where the expression is build from constants and the ring operations and the fractional and integer part operators. Following [2], we will call such expressions *bounded generalized polynomials*. The reason for performing this transformation is to relate our questions about diophantine correct rings to a substantial body of results about the distribution of the values of generalized polynomials.

PROPOSITION 3.4. Assume  $0 < r_i < 1$ . For each system of polynomial inequalities

$$\begin{aligned} |\sigma_1(y_0) - y_1 - r_1| &< \varepsilon \\ |\sigma_2(y_0, y_1) - y_2 - r_2| &< \varepsilon \\ &\dots\dots\dots \\ |\sigma_n(y_0, y_1 \dots y_{n-1}) - y_n - r_n| &< \varepsilon \end{aligned}$$

there is an associated system of bounded generalized polynomial inequalities

$$\bigwedge_{i=1}^n |\gamma_i(y_0) - r_i| < \varepsilon$$

where the  $\gamma_i$  are defined by

$$\gamma_i(y_0) = \sigma_i(y_0 \dots y_{i-1}) - y_i,$$

and the  $y_i$  for  $i > 0$  are defined recursively by

$$y_i = \lfloor \sigma_i(y_0 \dots y_{i-1}) \rfloor.$$



*Specifically,*

$$\begin{aligned}\gamma_1(y_0) &= \{\sigma_1(y_0)\} \\ \gamma_2(y_0) &= \{\sigma_2(y_0, \lfloor \sigma_1(y_0) \rfloor)\} \\ \gamma_3(y_0) &= \{\sigma_3(y_0, \lfloor \sigma_1(y_0) \rfloor, \lfloor \sigma_2(y_0, \lfloor \sigma_1(y_0) \rfloor) \rfloor)\} \\ &\dots\dots\end{aligned}$$

*For all sufficiently small  $\varepsilon > 0$  an integer  $y_0$  satisfies the associated system if and only if there are integers  $y_1 \dots y_n$  such that  $y_0 \dots y_n$  satisfies the original system.*

Since open induction is essentially the theory of abstract integer parts, there is an obvious connection between open induction and generalized polynomials, yet a systematic study of generalized polynomials *vis a vis* open induction remains to be done.

There are generalized polynomial identities, that hold for all integers, such as

$$\{\sqrt{2}x\}^2 = \{2\sqrt{2}x\{\sqrt{2}x\}\}.$$

Observe that this phenomenon can be explained by the fact that the ring

$$\mathbb{Z}[t, \sqrt{2}t - r, 2\sqrt{2}rt - s],$$

where  $r$  and  $s$  are algebraically independent real numbers, does not extend to a model of open induction. Indeed, we have the identity

$$H(t, \sqrt{2}t - r, 2\sqrt{2}rt - s) = s - r^2,$$

where  $H(x, y, z) = 2x^2 - y^2 - z$ ; so the ring is not discretely ordered. Substituting  $\{\sqrt{2}x\}$  for  $r$  and  $\{2\sqrt{2}x\{\sqrt{2}x\}\}$  for  $s$  one immediately deduces the generalized polynomial identity mentioned above.

Do all generalized polynomial identities arise in this way from ordered rings that violate open induction?

**Theorems on generalized polynomials.** The study of systems of polynomial inequalities of type

$$\begin{aligned}|\sigma_1(y_0) - y_1| &< \varepsilon \\ |\sigma_2(y_0, y_1) - y_2| &< \varepsilon \\ &\dots\dots \\ |\sigma_n(y_0, y_1 \dots y_{n-1}) - y_n| &< \varepsilon\end{aligned}\tag{*}$$

goes back at least to Van der Corput. He proved

**THEOREM 3.5** (Van der Corput [9]). *If a system of polynomial inequalities of type (\*) has a solution in integers then it has infinitely many integer solutions.*

Moreover, the set  $S \subseteq \mathbb{Z}$  of integers  $y_0$  for which there is a solution  $y_0 \dots y_n$  is syndetic.<sup>19</sup>

As far we know no one has given an algorithm for the solvability of arbitrary systems of type (\*). We believe that if a system of type (\*) with real algebraic coefficients has no integer solutions, then this fact is provable from the axioms of open induction.

By far the most far-reaching results on generalized polynomials are to be found in Bergelson and Leibman [2]. We paraphrase an important result from this paper, for use in Section 4.

**THEOREM 3.6** (Bergelson and Leibman [2]). *Let  $g : \mathbb{Z} \rightarrow \mathbb{R}^n$  be a map whose components are bounded generalized polynomials. Then there is a subset  $S$  of  $\mathbb{Z}$  of density<sup>20</sup> zero such that the closure of the set of values of  $g$  on the integers not in  $S$  is a semialgebraic set  $C$ . (I.e.  $C$  is definable by a formula with real parameters in the language of ordered rings.) If the coefficients of  $g$  are algebraic then  $C$  is definable without parameters.*

**§4. A Class of Diophantine correct ordered rings.** The next theorem identifies a class of diophantine correct ordered rings made from special sequences of polynomials.

**THEOREM 4.1.** *For  $i = 1 \dots n$  let  $g_i(t, x_1, \dots, x_{i-1})$  be polynomials with algebraic coefficients. For each  $n$ -tuple of algebraically independent real numbers  $\bar{r}$  such that  $0 < r_i < 1$ , let  $R_{\bar{r}}$  be the ring*

$$\mathbb{Z}[t, g_1(t) - r_1, g_2(t, r_1) - r_2, \dots, g_n(t, r_1, \dots, r_{n-1}) - r_n]$$

*Then*

- (1) *If the ring  $R_{\bar{r}}$  extends to a model of open induction for one algebraically independent  $n$ -tuple  $\bar{r}$  then it does so for all such  $n$ -tuples  $\bar{r}$ .*
- (2) *If the rings  $R_{\bar{r}}$  extend to models of open induction, then there is an open subset  $S$  of the unit box  $[0, 1]^n$  such that for all algebraically independent  $\bar{r} \in S$ , the ring  $R_{\bar{r}}$  is diophantine correct.*

**PROOF OF (1).** Let  $\bar{r}$  be an  $n$ -tuple of real numbers with algebraically independent coordinates. Since the ring  $R_{\bar{r}}$  is generated by algebraically independent polynomials,  $R_{\bar{r}}$  will extend to a model of open induction if and only if it is discretely ordered. (See the section on Wilkie's theorems in the Introduction.) If  $R_{\bar{r}}$  is not discretely ordered, then there is an identity of polynomials

<sup>19</sup>A subset  $S$  of  $\mathbb{Z}$  is syndetic if there are finitely many integers  $v_i \in \mathbb{Z}$  such that the union of translates  $\bigcup_i S + v_i$  is equal to  $\mathbb{Z}$ . Equivalently, the gaps between the elements of  $S$  have bounded lengths.

<sup>20</sup>Density means here Folner density, defined as follows. A Folner sequence (in  $\mathbb{Z}$ ) is a sequence of finite subsets  $s_i$  of  $\mathbb{Z}$  such that for every  $n \in \mathbb{Z}$ ,  $\lim_{m \rightarrow \infty} |(s_m + n) \Delta s_m| / |s_m| = 0$ . Here  $\Delta$  means symmetric difference, and  $s_m + n = \{x + n : x \in s_m\}$ . A set of integers  $S$  has Folner density zero if  $\lim_{n \rightarrow \infty} |S \cap s_n| / |s_n| = 0$  for every Folner sequence  $s_n$ .

in  $t$  of the form

$$H(t, g_1(t) - r_1, \dots, g_n(t, r_1 \dots r_{n-1}) - r_n) = K(\bar{r}),$$

where  $H$  and  $K$  are polynomials and  $H$  has integer coefficients. If such an identity holds for one tuple  $\bar{r}$  with algebraically independent coordinates, then it holds for them all.  $\dashv$

PROOF OF (2). The case  $n = 1$  is done in Example 2.2. We show there that one can take  $S$  to be the interval  $(0, 1)$ .

Assume  $n > 1$ , and assume that the rings  $R_{\bar{r}}$  extend to models of open induction. The proof will proceed by induction on  $n$ .

Copying Proposition 3.3, we construct a sequence of polynomials  $\sigma_i$  inductively as follows: Let  $\sigma_1(y_0) = f_1(y_0)$ . For  $i > 1$ , let

$$\sigma_i(y_0 \dots y_{i-1}) = g_i(y_0, \sigma_1(y_0) - y_1, \dots, \sigma_{i-1}(y_0 \dots y_{i-2}) - y_{i-1}).$$

Then the ring  $R_{\bar{r}}$  is diophantine correct if and only if for all positive  $\varepsilon$  the inequalities

$$\begin{aligned} |\sigma_1(y_0) - y_1 - r_1| &< \varepsilon \\ |\sigma_2(y_0, y_1) - y_2 - r_2| &< \varepsilon \\ &\dots\dots\dots \\ |\sigma_n(y_0, y_1 \dots y_{n-1}) - y_n - r_n| &< \varepsilon \end{aligned} \tag{*}$$

have integer solutions  $y_i$ . As in Proposition 3.4, we define the generalized polynomials

$$\gamma_i(y_0) = \sigma_i(y_0 \dots y_{i-1}) - y_i,$$

where  $y_i$  is defined inductively by

$$y_i = \lfloor \sigma_i(y_0 \dots y_{i-1}) \rfloor.$$

Then for small enough  $\varepsilon$  the inequalities  $|\gamma_i(y_0) - r_i| < \varepsilon$  hold for  $y_0$  if and only if the inequalities  $(*)$  hold for  $y_0$  and some choice of integers  $y_1 \dots y_n$ .

By Theorem 3.5, there is a subset  $B$  of  $\mathbb{Z}$  of Folner density 0 such that the closure of the points  $\bar{\gamma}(x)$  for  $x \notin B$  is a semialgebraic set  $C$  defined over  $\mathbb{Q}$ .

If the cell decomposition of  $C$  has an  $n$ -dimensional cell,<sup>21</sup> then  $C$  contains an open subset of  $[0, 1]^n$  and the theorem is proved. Otherwise, there is a non-zero polynomial  $h$  with integer coefficients such that

$$h(\gamma_1(x) \dots \gamma_n(x)) = 0$$

for all integers  $x$  not in  $B$ .<sup>22</sup>

Our goal is to prove that this is impossible, by showing that if such an equation held, then  $R_{\bar{r}}$  would not be discretely ordered.

<sup>21</sup>See [8] Chapter 3.

<sup>22</sup>Semialgebraic sets of codimension at least one satisfy nontrivial polynomial equations. [8]

By the induction hypothesis there is an open set  $S \subseteq [0, 1]^{n-1}$  such that for all points  $\bar{s} \in S$  with algebraically independent coordinates, the rings  $R_{\bar{s}}$  are diophantine correct.

Fix a point  $\bar{s} \in S$  with algebraically independent coordinates. We shall need to know that there are integers  $m \notin B$  for which the point  $(\gamma_1(m) \dots \gamma_{n-1}(m))$  comes arbitrarily close to  $\bar{s}$ .

Let  $\varepsilon > 0$ . Since  $R_{\bar{s}}$  is diophantine correct, Proposition 3.3 Part (1) and Proposition 3.4 imply that there is an integer  $m$  such that

$$|(\gamma_1(m) \dots \gamma_{n-1}(m)) - \bar{s}| < \varepsilon. \quad (**)$$

By Theorem 3.5 the solutions to  $(**)$  are syndetic. But no syndetic set has Folner density zero.<sup>23</sup> Therefore, for each  $\varepsilon > 0$  there is an integer  $m \notin B$  satisfying  $(**)$ .

Fix a sequence of integers  $m_i \notin B$  such that the point  $(\gamma_1(m_i) \dots \gamma_{n-1}(m_i))$  tends to  $\bar{s}$ .

Define  $V \subseteq \mathbb{Z}^{n+1}$  to be the set of all points

$$(m_i, \lfloor g_1(m_i) \rfloor, \lfloor g_2(m_i, \gamma_1(m_i)) \rfloor, \dots, \lfloor g_n(m_i, \gamma_1(m_i) \dots \gamma_{n-1}(m_i)) \rfloor)$$

for  $i = 1, 2, \dots$

The equation  $h(\gamma_1(m_i) \dots \gamma_{n-1}(m_i)) = 0$  holds for all  $i$ . Therefore, the equation

$$h(\sigma_1(y_0) - y_1 \dots \sigma_n(y_0 \dots y_{n-1}) - y_n) = 0$$

holds for all points  $(y_0 \dots y_n) \in V$ . Let  $H(\bar{y})$  denote the polynomial on the left of the above expression, so  $H(\bar{y})$  has algebraic coefficients and vanishes on  $V$ .

We claim that  $H$  must have a non-constant factor with rational coefficients. We shall prove this by arguing that the Zariski closure of  $V$  over the complex numbers includes a hypersurface in  $\mathbb{C}^{n+1}$ . The vanishing ideal of that hypersurface will be principal, and defined over  $\mathbb{Q}$ , hence generated by a rational polynomial. That rational polynomial will be a divisor of  $H$ .

To proceed, choose an infinite subset  $V_0$  of  $V$  such that the Zariski closure  $Z$  of  $V_0$  is an irreducible component of the Zariski closure of  $V$ . We will show that  $Z$  is a hypersurface in  $\mathbb{C}^{n+1}$  by arguing that no non-zero complex polynomial  $k(y_0 \dots y_{n-1})$  vanishes on  $V_0$ .

Just suppose that  $k(y_0 \dots y_{n-1})$  did vanish on  $V_0$ . Since  $V_0 \subset \mathbb{R}^{n+1}$ , we can assume that  $k$  has real coefficients. Since the coordinates of  $\bar{s}$  are algebraically independent, it follows that  $k(t, g_1(t) - s_1, \dots, g_{n-1}(t, s_1 \dots s_{n-2}) - s_{n-1})$  is

<sup>23</sup>Let  $s_i$  be the set of integers between  $-i$  and  $i$ . Then  $s_i$  is a Folner sequence. If  $D$  is any syndetic set of integers, then choose  $M$  so that  $D$  meets every interval of length  $M$ . Then  $\liminf_{i \rightarrow \infty} |D \cap s_i|/|s_i|$  will be at least  $1/M$ , so  $D$  cannot have density 0.

not the zero polynomial. Write

$$k(t, g_1(t) - s_1, \dots, g_{n-1}(t, s_1 \dots s_{n-2}) - s_{n-1}) = \sum_{i=1}^L k_i(\bar{s}) t^i$$

with  $k_L(\bar{s}) \neq 0$ . Choose a neighborhood  $U$  of  $\bar{s}$  on which  $k_L(\bar{x})$  is bounded away from zero. Then we can choose  $M$  so large that for  $t > M$  and for  $\bar{x} \in U$ , it holds that

$$k(t, g_1(t) - x_1, \dots, g_{n-1}(t, x_1 \dots x_{n-2}) - x_{n-1}) \neq 0. \quad (***)$$

Now choose  $i$  so that

- (1)  $m_i > M$ .
- (2)  $(\gamma_1(m_i) \dots \gamma_{n-1}(m_i)) \in U$ .
- (3)  $(m_i, \lfloor g_1(m_i) \rfloor, \lfloor g_2(m_i, \gamma_1(m_i)) \rfloor, \dots, \lfloor g_n(m_i, \gamma_1(m_i) \dots \gamma_{n-1}(m_i)) \rfloor) \in V_0$ .

Substituting  $\gamma_1(m_i) \dots \gamma_{n-1}(m_i)$  for  $x_1 \dots x_{n-1}$  and also  $m_i$  for  $t$  in (\*\*\*) we obtain

$$k(m_i, g_1(m_i) - \gamma_1(m_i), \dots, g_{n-1}(m_i, \gamma_1(m_i), \dots, \gamma_{n-2}(m_i)) - \gamma_{n-1}(m_i)) \neq 0.$$

Looking at the definition of the  $\gamma_i$ , we see that the above inequation is equivalent to

$$k(m_i, \lfloor g_1(m_i) \rfloor, \lfloor g_2(m_i, \gamma_1(m_i)) \rfloor, \dots, \lfloor g_n(m_i, \gamma_1(m_i), \dots, \gamma_{n-1}(m_i)) \rfloor) \neq 0.$$

But this is a contradiction, because the point

$$(m_i, \lfloor g_1(m_i) \rfloor, \lfloor g_2(m_i, \gamma_1(m_i)) \rfloor, \dots, \lfloor g_n(m_i, \gamma_1(m_i), \dots, \gamma_{n-1}(m_i)) \rfloor)$$

is an element of  $V_0$ , hence  $k$  vanishes at this point. We conclude that  $Z$ , which is the Zariski closure of  $V_0$ , is a hypersurface in  $\mathbb{C}^{n+1}$ .

The vanishing ideal  $I \subseteq \mathbb{C}[\bar{y}]$  of  $Z$  is principal. Since  $Z$  is the Zariski closure of a set of points with integer coordinates,  $I$  has a generator  $Q$  in  $\mathbb{Q}[\bar{y}]$ . The polynomial  $Q$  is the divisor of  $H$  that we were after.

To complete the proof, suppose  $H$  factors as  $Q \cdot P$ . Then the coefficients of  $P$  are real algebraic numbers, and we have the following equality of polynomials:

$$Q(\bar{y}) \cdot P(\bar{y}) = h(\sigma_1(y_0) - y_1 \dots \sigma_n(y_0 \dots y_{n-1}) - y_n)$$

Substituting  $g_i(t, \bar{s}) - r_i$  for  $y_i$  ( $i = 1 \dots n$ ) in the last equation, we obtain

$$A \cdot B = h(r_1 \dots r_n),$$

where

$$A = Q(t, g_1(t) - r_1, \dots, g_n(t, r_1 \dots r_{n-1}) - r_n)$$

and

$$B = P(t, g_1(t) - r_1, \dots, g_n(t, r_1 \dots r_{n-1}) - r_n).$$

Working in the ordered ring

$$\mathbb{Q}[t, g_1(t) - r_1, g_2(t, r_1) - r_2, \dots, g_n(t, r_1, \dots, r_{n-1}) - r_n],$$

we have that  $A \cdot B$  is finite, and neither is infinitesimal, therefore both are finite. But  $A$  has the form  $A_1/n$ , where  $A_1$  is a polynomial with integer coefficients. Thus  $A_1$  is a finite transcendental element of  $R_{\bar{r}}$ . But then  $R_{\bar{r}}$  is not discretely ordered, contrary to our assumption that  $R_{\bar{r}}$  extends to a model of open induction.  $\dashv$

REMARK 4.2. Theorem 4.1 is almost certainly not giving the whole truth. We believe that a ring  $R_{\bar{r}}$  generated by a special sequence is diophantine correct if and only if it extends to a model of open induction, with no restrictions on the tuple  $\bar{r}$  beyond algebraic independence. We also believe that a theorem like Theorem 4.1 holds for the more general sequences of Puiseux polynomials used to construct models of open induction in [1]. To prove this, one must extend the results of [2] to an appropriate class of “generalized” semialgebraic functions, that is, compositions of semialgebraic functions with the integer part operator.

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# TENNENBAUM'S THEOREM AND RECURSIVE REDUCTS

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*To honor and celebrate the memory of Stanley Tennenbaum*

Stanley Tennenbaum's influential 1959 theorem asserts that there are no recursive nonstandard models of Peano Arithmetic (PA). This theorem first appeared in his abstract [42]; he never published a complete proof. Tennenbaum's Theorem has been a source of inspiration for much additional work on nonrecursive models. Most of this effort has gone into generalizing and strengthening this theorem by trying to find the extent to which PA can be weakened to a subtheory and still have no recursive nonstandard models. Kaye's contribution [12] to this volume has more to say about this direction.

This paper is concerned with another line of investigation motivated by two refinements of Tennenbaum's theorem in which not just the model is nonrecursive, but its additive and multiplicative reducts are each nonrecursive. For the following stronger form of Tennenbaum's Theorem credit should also be given to Kreisel [5] for the additive reduct and to McAloon [26] for the multiplicative reduct.

**TENNENBAUM'S THEOREM.** *If  $\mathcal{M} = (M, +, \cdot, 0, 1, \leq)$  is a nonstandard model of PA, then neither its additive reduct  $(M, +)$  nor its multiplicative reduct  $(M, \cdot)$  is recursive.*

What happens with other reducts? The behavior of the order reduct, as is well known, is quite different from that of the additive and multiplicative reducts. The order type of every countable nonstandard model is  $\omega + (\omega^* + \omega) \cdot \eta$ , where  $\omega$  and  $\eta$  are the order types of the nonnegative integers  $\mathbb{N}$  and the rationals  $\mathbb{Q}$ , respectively. It is immediate from the definition of this order type that there is a recursive linearly ordered set having this order type. Thus, any attempt to adapt Tennenbaum's Theorem to order reducts is doomed. This essentially exhausts all the reducts, so in order to investigate further possible generalizations of Tennenbaum's Theorem, we will generalize the notion of a reduct.

Consider a tuple  $\bar{\varphi} = \varphi_0(\bar{x}), \varphi_1(\bar{x}), \dots, \varphi_k(\bar{x})$  of formulas in the language of PA, where  $\varphi_i(\bar{x})$  is an  $n_i$ -ary formula. Given a model  $\mathcal{M} \models \text{PA}$ , we define the *generalized reduct*  $\mathcal{M} \upharpoonright [\bar{\varphi}]$  to be the structure  $(M, R_0, R_1, \dots, R_k)$ , where  $R_i$  is the  $n_i$ -ary relation defined by  $\varphi_i(\bar{x})$ . Thus, the additive reduct  $(M, +)$  is

$\mathcal{M} \upharpoonright [x = y + z]$  and the multiplicative reduct  $(M, \cdot)$  is  $\mathcal{M} \upharpoonright [x = y \cdot z]$ . Let  $\text{Th}[\bar{\varphi}]$  be the theory of the class of all generalized reducts  $\mathcal{M} \upharpoonright [\bar{\varphi}]$ , where  $\mathcal{M}$  ranges over all models of PA. Then, Presburger Arithmetic (Pr) is  $\text{Th}[x = y + z]$  and Skolem Arithmetic (Sk) is  $\text{Th}[x = y \cdot z]$ <sup>1</sup>.

For various specific tuples  $\bar{\varphi}$ , the question of whether or not all generalized reducts  $\mathcal{M} \upharpoonright [\bar{\varphi}]$ , where  $M$  is a nonstandard model, are nonrecursive can be (and has been) asked. Two closely related generalized reducts were considered in [37]. Let  $|$  be the partial ordering of divisibility; that is, if  $a, b$  are elements of the nonstandard model  $M$ , then  $a|b$  iff  $a$  divides  $b$ . Then  $(M, |)$  is the *divisibility poset*, being the generalized reduct  $\mathcal{M} \upharpoonright [\exists z(y = x \cdot z)]$ . The *divisibility lattice* is the generalized reduct  $(M, \wedge, \vee) = \mathcal{M} \upharpoonright [z = \gcd(x, y), z = \text{lcm}(x, y)]$ . (We are ignoring the annoyance caused by 0.) Let  $\text{DP} = \text{Th}[\exists z(y = x \cdot z)]$  and  $\text{DL} = \text{Th}[z = \gcd(x, y), z = \text{lcm}(x, y)]$ .

I proved in [37] that there are recursively saturated (hence, nonstandard) models  $M$  of PA whose generalized reducts  $(M, |)$  and  $(M, \wedge, \vee)$  are recursive, despite the fact that both DP and DL are rich theories. (See Definition 1.1.) The proof of this will be presented in §4. It was noted in [37] that if  $M$  is a nonstandard model of True Arithmetic (TA), then its divisibility poset is not recursive; moreover, the same conclusion holds if  $M$  is 1-correct. (See Corollary 1.5.) These results are indicative of the theme of this paper. Given a tuple  $\bar{\varphi}$ , we ask if there are (recursively saturated) models  $M$  of PA for which  $\mathcal{M} \upharpoonright [\bar{\varphi}]$  is recursive, and, if so, how close to TA can we get  $\text{Th}(M)$  to be.

Following the preliminary §0 in which some of the notational and terminological conventions used in this paper are given, §1 presents the generalization of Tennenbaum's Theorem for generalized reducts. The key notions here are that of a rich theory and its hierarchal refinements,  $n$ -rich theories. Most of this paper is concerned with generalized reducts for which Tennenbaum's Theorem fails. The notions of a thin theory and its hierarchal refinements,  $n$ -thin theories, introduced in §2, capture the key ideas in showing this failure. The rest of this paper discusses various thin and  $n$ -thin theories. It is shown in §3 that the  $n$ -thin hierarchy does not collapse. Using well-quasi-ordered classes of finite structure, we show in §4 that there are many 1-thin theories, such as the theory of trees. The theory of linearly ordered sets is shown in §5 to be 2-thin. Finally, in §6, the results of §4 are used to show that a previously studied theory of tournaments is 1-thin.

The books [13] and [18] together provide a comprehensive treatment of models of Peano Arithmetic. I have been influenced by two classic papers on

<sup>1</sup>These two theories, which were first identified by Presburger [23] and Skolem [39], are usually defined as  $\text{Pr} = \text{Th}(\omega, +)$  and  $\text{Sk} = \text{Th}(\omega, \cdot)$ . It requires proof that  $\text{Pr} = \text{Th}[x = y + z]$  and  $\text{Sk} = \text{Th}[x = y \cdot z]$ . For Pr, this is a consequence of the quantifier-elimination result in [23]. For Sk it had been apparently overlooked, until proved by Nadel [29] and by Ciegelski [3], that this required proof.

Peano Arithmetic. Jensen and Ehrenfeucht [10] even today is an invaluable resource, and Smoryński [41] has much of interest concerning Tennenbaum's Theorem and some of its pre-1980's history. The paper [8] is a good source of information about recursive model theory.

**§0. Conventions.** We will typically use gothic letters such as  $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{B}$  to denote arbitrary structures. It is always to be understood that the universes of  $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{B}, \dots$  are  $A, A_1, B, \dots$ . However, for models of PA, we will use script letters such as  $\mathcal{M}, \mathcal{N}$  having universes  $M, N$ .

Although the usual formulation of PA is for the language  $\mathcal{L}_{\text{PA}} = \{+, \cdot, \leq, 0, 1\}$  that includes the two function symbols  $+$  and  $\cdot$ , we will adopt the convention that languages contain no function symbols, and that any symbol that appears to be a function symbol should be construed as a relation symbol in the most natural way. In particular  $+$  and  $\cdot$  are 3-ary relation symbols. This convention has no substantive effect, but it will be convenient to have at several places in this paper. Note that constant symbols are allowed.

In the literature, there are various ways that the notion of a (first-order) theory is formally defined, and for the most part it makes little or no difference which one is used. For us, it will be important to be definitive; we define a theory to be a consistent set of sentences. Thus, for example, PA is a recursive theory. If  $\mathcal{L}$  is a language and all sentences in the theory  $T$  are  $\mathcal{L}$ -sentences, then we will say that  $T$  is an  $\mathcal{L}$ -theory. An  $\mathcal{L}$ -theory is decidable if its deductive closure (that is, the smallest deductively closed  $\mathcal{L}$ -theory extending  $T$ ) is recursive. When we say that an  $\mathcal{L}$ -theory  $T$  is complete, we mean that for any  $\mathcal{L}$ -sentence  $\sigma$ , either  $\sigma \in T$  or  $\neg\sigma \in T$ .

There are two hierarchies of formulas that are used here. If  $n < \omega$ , then a formula in an arbitrary language is an  $\exists_n$  formula if it is in prenex form with  $n$  blocks of quantifiers beginning with a block of existential quantifiers. Define  $\forall_n$  analogously. Thus,  $\exists_0 = \forall_0$  are the quantifier-free formulas. We sometimes write  $\exists$  and  $\forall$  for  $\exists_1$  and  $\forall_1$ , respectively. Existential formulas are the same as  $\exists$  formulas, and universal formulas the same as  $\forall$  formulas. A *literal* is a sentence that is either an atomic sentence or the negation of an atomic sentence. We let  $\nabla$  be the set of literals.

In the case of the language  $\mathcal{L}_{\text{PA}}$ , we use the usual  $\Sigma_n, \Pi_n, \Delta_n$  to denote certain sets of formulas, but we restrict their use to  $n \geq 1$  since we are using  $+$  and  $\cdot$  as relation symbols. It is to be noted that by the MRDP Theorem (see [13]), if  $n \geq 1$  and  $\theta(\bar{x})$  is a  $\Sigma_n$  formula in the language of PA, then there is an  $\exists_n$  formula  $\varphi(\bar{x})$  such that  $\text{PA} \vdash \forall \bar{x} (\varphi(\bar{x}) \leftrightarrow \theta(\bar{x}))$ . An interesting sidebar is the theorem of Kaye [14] (also in [12]) that any recursive discretely ordered semiring satisfying some version of MRDP must be isomorphic to the standard model of PA.

We generally adhere to the convention that languages are finite, but we do make an exception in two situations in which infinitely many constant

symbols are allowed. One exception is that when constructing models using a Henkin construction, we allow for the introduction of infinitely many constant symbols. The second is a closely related situation. If  $\mathfrak{A}$  is a structure, then  $\mathcal{D}(\mathfrak{A})$  is the *complete* (or *elementary*) *diagram* of  $\mathfrak{A}$ ; that is,

$$\mathcal{D}(\mathfrak{A}) = \text{Th}((\mathfrak{A}, a)_{a \in A}).$$

Thus, as long as  $\mathfrak{A}$  is infinite, then  $\mathcal{D}(\mathfrak{A})$  is a theory for an infinite language. The *diagram* of  $\mathfrak{A}$  is  $\nabla \cap \mathcal{D}(\mathfrak{A})$ . The structure  $\mathfrak{A}$  is *decidable* if  $\mathcal{D}(\mathfrak{A})$  is recursive, and it is *recursive* if its diagram is recursive. We will sometimes identify a structure with its diagram. For example, when we say that  $\mathfrak{A}$  is recursive in  $X$ , then we mean that its diagram is recursive in  $X$ .

Notationally, we distinguish between  $\Sigma_n, \Pi_n, \Delta_n$  and  $\Sigma_n^0, \Pi_n^0, \Delta_n^0$ . As just mentioned, we use  $\Sigma_n, \Pi_n, \Delta_n$  to denote certain sets of formulas in the language of PA, and then we use (for example)  $\Sigma_n^0$  for the set of those subsets of  $\omega$  that are definable in the standard model by a formula in  $\Sigma_n$  or, possibly, for the set of those subsets of some tacitly understood set with a  $\Sigma_n^0$  set of Gödel numbers. If  $T$  is a theory, we sometimes say that  $T$  is *recursively axiomatizable* to mean that  $T$  is  $\Sigma_1^0$ .

**§1. Rich theories.** In discussing Tennenbaum's Theorem, Macintyre [24] points out “the conflict between *recursive* and *recursive saturation*” and credits Tennenbaum with being the first to call attention to this. I have taken this very suggestive phrase as the slogan for this section's presentation of a hierarchal generalization of Tennenbaum's Theorem for generalized reducts. The proof is not much more than a routine generalization of what is most likely the original proof (as, say, presented in [12]) of Tennenbaum's Theorem.

The notion of a rich theory plays a key role, either implicitly or explicitly, in many presentations of Tennenbaum's Theorem. To the best of my knowledge, the notion of a rich theory was first isolated and named in [10]. The term *effectively perfect* is used in [25] for an equivalent property. The following definition includes a hierarchal refinement of this notion.

**DEFINITION 1.1.** A theory  $T$  is *rich* if there are  $r < \omega$  and a recursive sequence  $\langle \theta_i(\bar{x}) : i < \omega \rangle$  of  $r$ -ary formulas such that whenever  $I, J \subseteq \omega$  are disjoint finite sets, then the sentence

$$\exists \bar{x} \left[ \bigwedge_{i \in I} \theta_i(\bar{x}) \wedge \bigwedge_{j \in J} \neg \theta_j(\bar{x}) \right]$$

is a consequence of  $T$ . If, in addition, each  $\theta_i(\bar{x})$  is an existential formula, then  $T$  is *existentially rich*. For a refinement of the notions of rich and existentially rich theories, if  $n < \omega$ , then  $T$  is *n-rich* if  $T$  is a rich theory and each  $\theta_i(\bar{x})$  in a sequence demonstrating that  $T$  is rich is in  $\Sigma_n$ .

Recall that we have conventionally assumed that languages are finite with no function symbols, so it follows that no theory is 0-rich.

The theory  $\text{Pr}$  is easily shown to be existentially rich by the sequence of formulas the  $i$ -th one of which asserts:  $x$  is a multiple of the  $i$ -th prime  $p_i$  (where  $p_0 = 2$ ). For example, if  $i = 2$  (so  $p_i = 5$ ), then the  $i$ -th formula is  $\exists y(y + y + y + y + y = x)$ . Similarly,  $\text{Sk}$  is shown to be existentially rich by a sequence whose  $i$ -th formula asserts:  $x$  is a  $p_i$ -th power.

Recall from the introduction that  $\text{DP}$  is the theory of the divisibility poset. The proof in [37] that  $\text{DP}$  is rich actually shows that it is 2-rich (although only 3-richness was claimed in [37]). The  $i$ -th formula  $\theta_i(x)$  in the sequence demonstrating this is:

$$\exists y_0, y_1, \dots, y_i, y_{i+1} \left[ \bigwedge_{j \leq i} (y_j | y_{j+1} \wedge y_j \neq y_{j+1}) \wedge y_i | x \wedge y_{i+1} \nmid x \right. \\ \left. \wedge \forall z \left( z | y_{i+1} \rightarrow \bigvee_{j \leq i+1} z = y_j \right) \right].$$

This sentence asserts: there is a prime  $y$  such that  $y^i | x$  and  $y^{i+1} \nmid x$ .

The reader is reminded that if  $\mathcal{M}$  is nonstandard, then its standard system  $\text{SSy}(\mathcal{M})$  is the set of all  $X \subseteq \omega$  that are coded in  $\mathcal{M}$ .

**THEOREM 1.2.** *Suppose that  $\bar{\varphi} = \varphi_0(\bar{x}), \varphi_1(\bar{x}), \dots, \varphi_k(\bar{x})$  is a tuple of formulas and  $\text{Th}[\bar{\varphi}]$  is  $(n+1)$ -rich. If  $\mathcal{M}$  is nonstandard and  $\emptyset^{(n)} \in \text{SSy}(\mathcal{M})$ , then  $\mathcal{M} \upharpoonright [\bar{\varphi}]$  is not recursive.*

The proof of this theorem will be broken up into two lemmas, each involving recursive saturation. Although this notion has a hazy history<sup>2</sup>, it is clear that its first explicit appearance in the model theory of PA was in [1]. A thorough introduction to recursive saturation can be found in [13]. An  $\mathcal{L}$ -structure  $\mathfrak{A}$  is *recursively saturated* if whenever a recursive set  $\Phi(\bar{x}, \bar{y})$  of  $(m+k)$ -ary  $\mathcal{L}$ -formulas  $\phi(\bar{x}, \bar{y})$  and a  $k$ -tuple  $\bar{a}$  from  $A$  are such that  $\Phi(\bar{x}, \bar{a})$  is finitely satisfiable in  $\mathfrak{A}$ , then  $\Phi(\bar{x}, \bar{a})$  is satisfiable in  $\mathfrak{A}$ . We give two generalizations of recursive saturation.

For the first one, if  $n < \omega$  and  $\Phi(\bar{x}, \bar{y})$  is restricted to being a set of formulas in  $\exists_n \cup \forall_n$ , then  $\mathfrak{A}$  is  *$n$ -restricted recursively saturated*. If  $\mathfrak{A}$  is  *$n$ -restricted recursively saturated* for all  $n < \omega$ , then  $\mathfrak{A}$  is *restricted recursively saturated*.

It is a fundamental fact that all nonstandard models of PA are restricted recursively saturated implying that all generalized reducts of a nonstandard model are restricted recursively saturated.

The second generalization of recursive saturation relativizes it to an arbitrary Turing ideal  $\mathfrak{X} \subseteq \mathcal{P}(\omega)$  (meaning: whenever  $X_1, X_2, \dots, X_k \in \mathfrak{X}$  and

<sup>2</sup>Check the historical remarks in [2, §3] and the somewhat idiosyncratic historical remarks in [40].

$X_0$  is Turing reducible to  $X_1 \times X_2 \times \cdots \times X_k$ , then  $X_0 \in \mathfrak{X}$ ). If  $n < \omega$ , then  $\Delta_{n+1}^0$  is a Turing ideal. If  $\mathfrak{X}$  is a Turing ideal, then  $\varnothing^{(n)} \in \mathfrak{X}$  iff  $\Delta_{n+1}^0 \subseteq \mathfrak{X}$ . In particular, every Turing ideal has  $\Delta_1^0$  as a subset. Another example of a Turing ideal is important for us: if  $\mathcal{M}$  is nonstandard, then  $\text{SSy}(\mathcal{M})$  is a Turing ideal. More generally, if  $\mathfrak{X} \subseteq \mathcal{P}(\omega)$ , then an  $\mathcal{L}$ -structure  $\mathfrak{A}$  is  $\mathfrak{X}$ -recursively saturated<sup>3</sup> if whenever  $\Phi(\bar{x}, \bar{y})$  is a set of  $(m+k)$ -ary  $\mathcal{L}$ -formulas that is recursive in some  $X \in \mathfrak{X}$ ,  $\bar{a}$  is a  $k$ -tuple from  $A$ , and  $\Phi(\bar{x}, \bar{a})$  is finitely satisfiable in  $\mathfrak{A}$ , then  $\Phi(\bar{x}, \bar{a})$  is satisfiable in  $\mathfrak{A}$ . In particular, recursive saturation is the same as  $\Delta_1^0$ -recursive saturation. Also,  $\Sigma_n^0$ -recursive saturation is the same as  $\Delta_n^0$ -recursive saturation, which is also the same as  $\{\varnothing^{(n-1)}\}$ -recursive saturation.

A basic fact is that every recursively saturated model  $\mathcal{M}$  of PA is  $\text{SSy}(\mathcal{M})$ -recursively saturated.

These two generalizations of recursive saturation can easily be amalgamated into one:  $\mathfrak{A}$  is  $n$ -restricted  $\mathfrak{X}$ -recursively saturated if whenever  $\Phi(\bar{x}, \bar{y}) \in \mathfrak{X}$  is a set of  $(m+k)$ -ary  $\mathcal{L}$ -formulas in  $\exists_n \cup \forall_n$ ,  $\bar{a}$  is a  $k$ -tuple from  $A$ , and  $\Phi(\bar{x}, \bar{a})$  is finitely satisfiable in  $\mathfrak{A}$ , then  $\Phi(\bar{x}, \bar{a})$  is satisfiable in  $\mathfrak{A}$ . If  $\mathfrak{A}$  is  $n$ -restricted  $\mathfrak{X}$ -recursively saturated for all  $n < \omega$ , then  $\mathfrak{A}$  is *restricted*  $\mathfrak{X}$ -recursively saturated.

It is easy to see that every nonstandard  $\mathcal{M}$  is restricted  $\text{SSy}(\mathcal{M})$ -recursively saturated. From this, the following is an immediate consequence.

LEMMA 1.3. *If  $\bar{\varphi} = \varphi_0(\bar{x}), \varphi_1(\bar{x}), \dots, \varphi_k(\bar{x})$  is a tuple of formulas and  $\mathcal{M}$  is nonstandard, then  $\mathcal{M} \upharpoonright [\bar{\varphi}]$  is restricted  $\text{SSy}(\mathcal{M})$ -recursively saturated.*

PROOF. Let  $\mathcal{L} = \{R_0, R_1, \dots, R_k\}$  be the language of  $\mathcal{M} \upharpoonright [\bar{\varphi}]$ , and let  $n < \omega$  be such that each  $\varphi_i(\bar{x})$  is in  $\exists_n$ . It suffices to observe that by substituting  $\varphi_i$  for  $R_i$ , we can get a recursive function  $\theta(\bar{y}) \mapsto \theta^*(\bar{y})$  mapping an  $\mathcal{L}$ -formula  $\theta(\bar{y})$  in  $\exists_m$  to an  $\mathcal{L}_{\text{PA}}$ -formula  $\theta^*(\bar{y})$  in  $\exists_{m+n}$  such that for any  $\bar{a}$ ,  $\mathcal{M} \upharpoonright [\bar{\varphi}] \models \theta(\bar{a}) \iff \mathcal{M} \models \theta^*(\bar{a})$ .  $\dashv$

LEMMA 1.4. *Let  $n < \omega$ . Suppose that  $T$  is an  $(n+1)$ -rich theory and that  $\mathfrak{A} \models T$  is  $(n+1)$ -restricted  $\Delta_{n+1}^0$ -recursively saturated. Then,  $\mathfrak{A}$  is not recursive.*

PROOF. Assume, for a contradiction, that  $\mathfrak{A}$  is recursive.

Let  $\langle \theta_i(\bar{x}) : i < \omega \rangle$  be a sequence of  $r$ -ary formulas demonstrating that  $T$  is  $(n+1)$ -rich. Let  $X, Y \subseteq \omega$  be disjoint  $\Sigma_{n+1}^0$  sets that are  $\Delta_{n+1}^0$ -inseparable; that is, there is no  $Z \in \Delta_{n+1}^0$  such that  $X \subseteq Z$  and  $Y \cap Z = \emptyset$ . Let  $\Phi(\bar{x}, \bar{y})$  be the set

$$\{\theta_i(\bar{x}) : i \in X\} \cup \{\theta_j(\bar{y}) : j \in Y\} \cup \{\theta_i(\bar{x}) \leftrightarrow \neg\theta_i(\bar{y}) : i < \omega\}.$$

Clearly,  $\Phi$  is a finitely satisfiable  $\Sigma_{n+1}^0$  set of formulas each of which is a conjunction of formulas in  $\exists_{n+1} \cup \forall_{n+1}$ . Thus, there are  $\bar{c}, \bar{d}$  in  $A$  such that  $\mathfrak{A} \models \Phi(\bar{c}, \bar{d})$ . Now let  $Z = \{i < \omega : \mathfrak{A} \models \theta_i(\bar{c})\} = \{i < \omega : \mathfrak{A} \models \neg\theta_i(\bar{d})\}$ .

<sup>3</sup>I would have preferred to use the simpler term  $\mathfrak{X}$ -saturated, but that already has an established meaning (as in [18]). I also considered using  $\mathfrak{X}$ -ly saturated, but was gently dissuaded from doing so.

Clearly,  $X \subseteq Z$  and  $Y \cap Z = \emptyset$ . Since  $\mathfrak{A}$  is recursive, then  $Z$  is  $\Delta_{n+1}^0$ , contradicting that  $X, Y$  are  $\Delta_{n+1}^0$ -inseparable.  $\neg$

Theorem 1.2 now follows. For, under its hypotheses,  $\mathcal{M} \upharpoonright [\bar{\varphi}]$  is restricted  $\Delta_{n+1}^0$ -recursively saturated by Lemma 1.3, and then it is not recursive by Lemma 1.4.

Tennenbaum's theorem as stated in the introduction is a consequence of this theorem, as is the next corollary.

**COROLLARY 1.5.** *If  $\mathcal{M}$  is nonstandard and has a recursive divisibility poset, then  $\emptyset' \notin \text{SSy}(\mathcal{M})$ .*  $\neg$

Theorem 1.2 can easily be relativized.

**COROLLARY 1.6.** *Suppose  $\bar{\varphi} = \varphi_0(\bar{x}), \varphi_1(\bar{x}), \dots, \varphi_k(\bar{x})$  is a tuple of formulas and  $\text{Th}[\bar{\varphi}]$  is  $(n+1)$ -rich. If  $\mathcal{M}$  is nonstandard and  $X^{(n)} \in \text{SSy}(\mathcal{M})$ , then  $\mathcal{M} \upharpoonright [\bar{\varphi}]$  is not recursive in  $X$ .*  $\neg$

Notice that the concluding sentence of the previous corollary is equivalent to:  $\mathcal{M} \upharpoonright [\bar{\varphi}]^{(n)} \notin \text{SSy}(\mathcal{M})$ . The next theorem, whose easy proof is an adaptation of the proof of Theorem 1.2, will be omitted.

**THEOREM 1.7.** *Suppose  $\bar{\varphi} = \varphi_0(\bar{x}), \varphi_1(\bar{x}), \dots, \varphi_k(\bar{x})$  is a tuple of formulas and  $\text{Th}[\bar{\varphi}]$  is rich. If  $\mathcal{M}$  is nonstandard and  $\mathcal{M} \upharpoonright [\bar{\varphi}]$  is recursively saturated, then  $\mathcal{M} \upharpoonright [\bar{\varphi}] \notin \text{SSy}(\mathcal{M})$ .*

In the previous theorem, it is required that  $\mathcal{M} \upharpoonright [\bar{\varphi}]$  be recursively saturated. There are two ways this may come about. First, if  $\mathcal{M}$  is recursively saturated, then so is  $\mathcal{M} \upharpoonright [\bar{\varphi}]$ . In fact, in parallel with Lemma 1.3, if  $\mathcal{M}$  is recursively saturated, then  $\mathcal{M} \upharpoonright [\bar{\varphi}]$  is  $\text{SSy}(\mathcal{M})$ -recursively saturated. The second way is that it might happen that  $\mathcal{M} \upharpoonright [\bar{\varphi}]$  is recursively saturated whenever  $\mathcal{M}$  is nonstandard. This is the case for the additive and multiplicative reducts and, hence, also for the divisibility posets.

Let TA be True Arithmetic, the theory of the standard model of PA. If  $\mathcal{M}$  is a nonstandard model of TA, then  $\mathcal{M}$  is  $n$ -correct for all  $n < \omega$ , so all arithmetic sets are in  $\text{SSy}(\mathcal{M})$ . Thus, the following is a consequence of Theorem 1.7.

**COROLLARY 1.8.** *Suppose  $\bar{\varphi} = \varphi_0(\bar{x}), \varphi_1(\bar{x}), \dots, \varphi_k(\bar{x})$  is a tuple of formulas and  $\text{Th}[\bar{\varphi}]$  is rich. If  $\mathcal{M}$  is a recursively saturated model of TA, then  $\mathcal{M} \upharpoonright [\bar{\varphi}]$  is not arithmetic.*  $\neg$

A special case of both Corollary 1.6 and of Theorem 1.7 is: *If  $\mathcal{M}$  is a nonstandard model of TA, then neither its additive reduct  $(\mathcal{M}, +)$  nor its multiplicative reduct  $(\mathcal{M}, \cdot)$  is arithmetic.* With the weaker conclusion that  $\mathcal{M}$  is not arithmetic, this is a theorem of Feferman [7] that predates Tennenbaum's Theorem.

**§2. Thin theories.** The generalizations of Tennenbaum's Theorem that were presented in the previous section required, at the least, that  $\text{Th}[\bar{\varphi}]$  be rich. If  $\text{Th}[\bar{\varphi}]$  is not rich, then there may be a nonstandard model  $\mathcal{M}$  such that  $\mathcal{M} \upharpoonright [\bar{\varphi}]$  is recursive. How do we find such a model? One almost trivial approach,

presented in Proposition 2.1, is to show that  $\text{Th}[\bar{\varphi}]$  has a recursive, recursively saturated model. This leads naturally to the notion of a thin theory and then to its hierarchal refinements. These refinements are our real interest here, although the definition of a thin theory is also included in Definition 2.2.

**PROPOSITION 2.1.** *Suppose that  $\bar{\varphi} = \varphi_0(\bar{x}), \varphi_1(\bar{x}), \dots, \varphi_k(\bar{x})$  is a tuple of formulas, and suppose that  $\text{Th}[\bar{\varphi}]$  has a recursive, recursively saturated model. Then there is a recursively saturated (and, hence, nonstandard) model  $\mathcal{M} \models \text{PA}$  such that  $\mathcal{M} \upharpoonright [\bar{\varphi}]$  is recursive.*

For the reader who knows about resplendent models, this proof is an easy one-liner. The following remarks are for the benefit of those who are not familiar with resplendent models. A more complete discussion can be found in [13].

An  $\mathcal{L}$ -structure  $\mathfrak{A}$  is *resplendent* if, whenever  $a_0, a_1, \dots, a_{n-1} \in A$ ,  $\mathcal{L}' \supseteq \mathcal{L}$  is a finite language,  $\Phi(\bar{x})$  is a recursive set of  $n$ -ary  $\mathcal{L}'$ -formulas, and  $\mathfrak{B} \succ \mathfrak{A}$  has an expansion  $\mathfrak{B}'$  such that  $(\mathfrak{B}', \bar{a}) \models \Phi(\bar{a})$ , then there is an expansion  $\mathfrak{A}'$  of  $\mathfrak{A}$  such that  $(\mathfrak{A}', \bar{a}) \models \Phi(\bar{a})$ . The crucial fact is that a countable structure is resplendent iff it is recursively saturated. Moreover, every countable resplendent structure  $\mathfrak{A}$  is *chronically* resplendent, which means that the expansion  $\mathfrak{A}'$  in the definition of resplendency can itself be resplendent.

**PROOF.** Let  $\mathfrak{A} \models \text{Th}[\bar{\varphi}]$  be a recursive, recursively saturated model. By the chronic resplendency of  $\mathfrak{A}$ , it can be expanded to a recursively saturated model  $\mathcal{M} \models \text{PA}$  such that  $\mathcal{M} \upharpoonright [\bar{\varphi}] = \mathfrak{A}$ .  $\dashv$

The proof of the previous proposition does not give much additional information about the model  $\mathcal{M}$ . The model is recursively saturated, but very little can be inferred about its theory  $\text{Th}(\mathcal{M})$  or its standard system  $\text{SSy}(\mathcal{M})$ . (Recall that a countable, recursively saturated model  $\mathcal{M}$  is determined up to isomorphism by  $\text{Th}(\mathcal{M})$  and  $\text{SSy}(\mathcal{M})$ .) Certainly, there is no reason to expect that  $\emptyset' \in \text{SSy}(\mathcal{M})$  as would be the case if  $\mathcal{M}$  were 1-correct. In fact, examples in §3 show that there are cases in which it must be that  $\emptyset' \notin \text{SSy}(\mathcal{M})$ .

Proposition 2.1 suggests the following definitions of a thin theory and of a 1-thin theory. The hierarchal refinements are obvious generalizations.

**DEFINITION 2.2.** Let  $T$  be an  $\mathcal{L}$ -theory. We say that  $T$  is *thin* if whenever  $\mathcal{L}' \supseteq \mathcal{L}$  is a finite language and  $T' \supseteq T$  is an  $\mathcal{L}'$ -theory, then  $T'$  has a model  $\mathfrak{A}'$  whose reduct  $\mathfrak{A} = \mathfrak{A}' \upharpoonright \mathcal{L}$  is recursive. If  $1 \leq n < \omega$ , then we say that  $T$  is *n-thin* if whenever  $\mathcal{L}' \supseteq \mathcal{L}$  is a finite language,  $T'$  is a  $\Sigma_n^0$  set of  $\mathcal{L}'$ -sentences such that  $T \cup T'$  is consistent, then  $T'$  has a model  $\mathfrak{A}'$  whose reduct  $\mathfrak{A} = \mathfrak{A}' \upharpoonright \mathcal{L}$  is recursive. We conventionally assume that  $T$  is 0-thin.

In the case of 1-thinness, a trick of Kleene<sup>4</sup> can be used to simplify the definition: an  $\mathcal{L}$ -theory  $T$  is 1-thin iff whenever  $\mathcal{L}' \supseteq \mathcal{L}$  is a language and  $\sigma$

<sup>4</sup>Kleene [16] observed that if  $T$  is a recursively axiomatizable  $\mathcal{L}$ -theory, then there are sentence  $\sigma$  for a language possibly larger than  $\mathcal{L}$  such that, for any infinite  $\mathcal{L}$ -structure  $\mathfrak{A}$ ,  $\mathfrak{A} \models T$  iff  $\mathfrak{A}$  has an expansion to some  $\mathfrak{A}' \models \sigma$ . Also, see [13, Theorem 15.11]. The earliest reference I have found in which this is called “a trick of Kleene” is [34].

an  $\mathcal{L}'$ -sentence such that  $T \cup \{\sigma\}$  is consistent, then  $T \cup \{\sigma\}$  has a model  $\mathfrak{A}'$  whose reduct  $\mathfrak{A}' \upharpoonright \mathcal{L}$  is recursive.

An even more general definition can be made. If  $X \subseteq \omega$ , then say that an  $\mathcal{L}$ -theory  $T$  is  $X$ -thin if whenever  $\mathcal{L}' \supseteq \mathcal{L}$  is a finite language,  $T'$  is a set of  $\mathcal{L}'$ -sentences such that  $T'$  is recursive enumerable in  $X$  and  $T \cup T'$  is consistent, then  $T \cup T'$  has a model  $\mathfrak{A}'$  whose reduct  $\mathfrak{A}' \upharpoonright \mathcal{L}$  is recursive. (Notice that it would have made no difference if we had required that  $T'$  be recursive, rather than recursively enumerable, in  $X$ .) Thus, if  $1 \leq n < \omega$ , then  $T$  is  $n$ -thin iff it is  $\emptyset^{(n-1)}$ -thin. We will not make any use of the more general notion of an  $X$ -thin theory except in making the proof of Corollary 2.4(1) a little quicker.

The next lemma implies that Definition 2.2 yields, as a bonus, that  $\mathfrak{A}$  (and even  $\mathfrak{A}'$ ) can be recursively saturated. But even a little more is true. Recall the notion of  $\aleph$ -recursive saturation from the previous section.

**LEMMA 2.3.** *Let  $1 \leq n < \omega$ , and let  $T$  be an  $n$ -thin  $\mathcal{L}$ -theory. If  $\mathcal{L}' \supseteq \mathcal{L}$  is a finite language and  $T'$  is a  $\Sigma_n^0$  set of  $\mathcal{L}'$ -sentences such that  $T \cup T'$  is consistent, then  $T \cup T'$  has a  $\Delta_n^0$ -recursively saturated model  $\mathfrak{A}'$  such that  $\mathfrak{A} = \mathfrak{A}' \upharpoonright \mathcal{L}$  is recursive.*

**PROOF.** Without any more trouble, we will suppose that  $X \subseteq \omega$  and that  $T$  is an  $X$ -thin theory, and then prove:

*If  $\mathcal{L}' \supseteq \mathcal{L}$  is a finite language and  $T$  is a set of  $\mathcal{L}'$ -sentences such that  $T'$  is recursive in  $X$  and  $T \cup T'$  is consistent, then  $T \cup T'$  has an  $\{X\}$ -recursively saturated model  $\mathfrak{A}'$  such that  $\mathfrak{A} = \mathfrak{A}' \upharpoonright \mathcal{L}$  is recursive.*

There are many ways one can prove this. We will give a rather sketchy one involving models of PA. With the aid of [13] if needed, the reader should be able to fill in details.

Let  $\mathcal{L}'' = \mathcal{L}' \cup \mathcal{L}_{\text{PA}} \cup \{c, U, S\}$ , where  $\mathcal{L}_{\text{PA}} = \{+, \cdot, 0, 1, \leq\}$ ,  $c$  is a constant symbol,  $U$  is a unary relation symbol, and  $S$  a 2-ary relation symbol. Let  $T''$  be a set of  $\mathcal{L}''$ -sentences such that  $T'' \supseteq T'$  and whenever  $\mathfrak{A}''$  is an  $\mathcal{L}''$ -structure and  $\mathcal{M} = \mathfrak{A}'' \upharpoonright \mathcal{L}_{\text{PA}}$ , then  $\mathfrak{A}'' \models T''$  iff the following hold:

- (1)  $\mathcal{M} \models \text{PA}$ ;
- (2)  $S$  is a partial nonstandard inductive satisfaction class for  $\mathcal{M}$ ;
- (3)  $c$  codes the set  $X \in \text{SSy}(\mathcal{M})$ ;
- (4) each  $R \in \mathcal{L}'$  denotes a  $\Delta_2$ -definable subset of  $\mathcal{M}$ ;
- (5)  $U$  denotes a  $\Delta_2$ -definable subset of  $\mathcal{M}$  that defines in  $\mathcal{M}$  the complete diagram of the  $\mathcal{L}'$ -structure  $\mathfrak{A}'' \upharpoonright \mathcal{L}'$ .

Refer to [13, §15.1] for an explanation of (2), to [13, p. 13] for (3), and to [13, §13.2] for (5).

Clearly, there is such a  $T''$  that is recursive in  $X$ . By the Arithmeticized Completeness Theorem [13, §13.2] and the existence of models having satisfaction classes [13, Proposition 15.1],  $T \cup T' \cup T''$  is consistent. Therefore,

since  $T$  is  $X$ -thin,  $T \cup T' \cup T''$  has a model  $\mathfrak{A}''$  whose reduct  $\mathfrak{A}'' \upharpoonright \mathcal{L}$  is recursive. Clearly,  $\mathfrak{A}' = \mathfrak{A}'' \upharpoonright \mathcal{L}'$  is a model of  $T'$ . Let  $\mathcal{M} = \mathfrak{A}'' \upharpoonright \mathcal{L}_{\text{PA}}$ . By [13, Proposition 15.4], it follows from (2) that  $\mathcal{M}$  is recursively saturated, and then from (3) that it is  $\{X\}$ -recursively saturated. It then follows from (5) that  $\mathfrak{A}'$  is also  $\{X\}$ -recursively saturated.  $\dashv$

Perhaps the following characterizations of thinness and  $n$ -thinness are more attractive than Definition 2.2.

**COROLLARY 2.4.** *Let  $T$  be an  $\mathcal{L}$ -theory.*

- (1)  *$T$  is thin iff every completion of  $T$  has a recursive,  $\aleph_0$ -saturated model.*
- (2) *If  $1 \leq n < \omega$ , then  $T$  is  $n$ -thin iff for every  $\Sigma_n^0$  set of  $\mathcal{L}$ -sentences such that  $T \cup T'$  is consistent,  $T \cup T'$  has a recursive,  $\Delta_n^0$ -recursively saturated model.*

**PROOF.** We first prove the following stronger version of (2):

*Suppose  $X \subseteq \omega$  and  $T$  is an  $\mathcal{L}$ -theory. Then,  $T$  is  $X$ -thin iff for every  $\mathcal{L}$ -theory  $T'$  such that  $T'$  is recursive in  $X$  and  $T \cup T'$  is consistent,  $T \cup T'$  has a recursive,  $\{X\}$ -recursively saturated model.*

If  $T$  is  $X$ -thin, then the proof of Lemma 2.3 yields the conclusion. For the converse, let  $\mathcal{L}' \supseteq \mathcal{L}$  be finite, and let  $T'$  be a set of  $\mathcal{L}'$ -sentences such that  $T'$  is recursive in  $X$  and  $T \cup T'$  is consistent. Let  $T''$  be the set of  $\mathcal{L}$ -sentences that are consequences of  $T'$ . Then  $T''$  is  $\Sigma_1^0$  relative to  $X$  and  $T \cup T''$  is consistent. Therefore,  $T \cup T''$  has a recursive,  $\{X\}$ -recursively saturated model  $\mathfrak{A}$ . Then,  $\mathfrak{A}$  can be expanded to a model  $\mathfrak{A}'$  of  $T'$ . (Here, we are invoking the basic fact ([13, Theorem 15.8], for example) that countable, recursively saturated models are resplendent. Actually, we are using the obvious relativization of this fact; the proof as given in [13] easily relativizes.)

Now we prove (1). The theory  $T$  is thin iff it is  $X$ -thin for every  $X \subseteq \omega$ . Notice that a structure is  $\aleph_0$ -saturated iff it is  $\mathcal{P}(\omega)$ -recursively saturated. Thus, if every completion of  $T$  has a recursive,  $\aleph_0$ -saturated model, then the proof of Lemma 2.3 implies that  $T$  is  $X$ -thin for each  $X \subseteq \omega$  and, therefore, is thin.

For the converse, suppose that  $T$  is thin. Every complete  $n$ -type containing  $T$  is realized in a recursive model. Since there are only countably many recursive models, there are only countably many complete  $n$ -types. Let  $X \subseteq \omega$  be such that every complete  $n$ -type is recursive in  $X$ . Let  $T'$  be a completion of  $T$  and then let  $\mathfrak{A}$  be a recursive,  $\{X\}$ -recursively saturated model of  $T'$ . Then,  $\mathfrak{A}$  is  $\omega$ -homogeneous (by [13, Corollary 15.16]) and realizes all types, so it is  $\aleph_0$ -saturated.  $\dashv$

A statement very closely related to the particular instance of the following lemma when  $n = 2$  has been previously considered (see [15, p. 180]) in an entirely different context.

**LEMMA 2.5.** *If  $T$  is  $n$ -rich, then  $T$  is not  $n$ -thin.*

PROOF. Let  $T$  be an  $\mathcal{L}$ -theory. The case when  $n = 0$  is vacuously true, so let  $n \geq 1$ . Suppose  $T$  is  $n$ -rich and  $n$ -thin. Let  $\langle \theta_i(\bar{x}) : i < \omega \rangle$  be a sequence of  $r$ -ary  $\exists_n$  formulas demonstrating that  $T$  is  $n$ -rich. Let  $X, Y \subseteq \omega$  be disjoint  $\Sigma_n^0$  sets that are  $\Delta_n^0$ -inseparable. Get  $\mathcal{L}'$  by adjoining to  $\mathcal{L}$  the  $2r$  new constant symbols  $c_0, c_1, \dots, c_{r-1}, d_0, d_1, \dots, d_{r-1}$ , and then let

$$T' = \{\theta_i(\bar{c}) : i \in X\} \cup \{\theta_j(\bar{d}) : j \in Y\} \cup \{\theta_i(\bar{c}) \leftrightarrow \neg\theta_i(\bar{d}) : i < \omega\},$$

which is  $\Sigma_n$ . Then  $T \cup T'$  is a theory. Since  $T$  is  $n$ -thin, we can let  $\mathfrak{A}' \models T \cup T'$  be such that  $\mathfrak{A} = \mathfrak{A}' \upharpoonright \mathcal{L}$  is recursive. Then

$$Z = \{i < \omega : \mathfrak{A} \models \theta_i(\bar{c})\} = \{i < \omega : \mathfrak{A} \models \neg\theta_i(\bar{d})\}$$

is  $\Delta_n^0$ . Since  $X \subseteq Z$  and  $Y \cap Z = \emptyset$ , this contradicts the  $\Delta_n^0$ -inseparability, and the lemma is proved.  $\dashv$

The next corollary indicates why the notion of  $n$ -thinness is useful when studying generalizations of Tennenbaum's Theorem.

COROLLARY 2.6. *Suppose  $\bar{\varphi} = \varphi_0(\bar{x}), \varphi_1(\bar{x}), \dots, \varphi_k(\bar{x})$  is a tuple of formulas and  $\text{Th}[\bar{\varphi}]$  is  $(n+1)$ -thin. Then there is a  $\Delta_{n+1}^0$ -recursively saturated,  $n$ -correct model  $\mathcal{M} \models \text{PA}$  such that  $\mathcal{M} \upharpoonright [\bar{\varphi}]$  is recursive.*

PROOF. Let  $T = \text{Th}[\bar{\varphi}]$ ,  $\mathcal{L}' = \mathcal{L} \cup \{+, \cdot, 0, 1, \leq\}$ , and let

$$T' = (\text{TA} \cap \Sigma_{n+1}) \cup \{\forall \bar{x} [R_i(\bar{x}) \leftrightarrow \varphi_i(\bar{x})] : i \leq k\}.$$

Apply Lemma 2.3.  $\dashv$

In general, the converse to the previous corollary is false. For a counterexample, let  $\varphi(x, y, z)$  be the formula  $x + y = z \rightarrow \text{Con}(\text{PA})$ . If  $\mathcal{M} = (M, \dots)$  is a nonstandard model of PA and  $M$  is a recursive subset of  $\omega$ , then  $\mathcal{M} \upharpoonright [\varphi]$  is recursive iff  $\mathcal{M} \models \text{Con}(\text{PA})$ . Thus,  $\text{Th} \upharpoonright [\varphi]$  has a recursive, recursively saturated model even though it is not 1-thin.

There are various relativizations of Corollary 2.6 that are true. We mention one possibility here. If  $\bar{\varphi}$  is a tuple of formulas and  $T \supseteq \text{PA}$  is a theory, then let  $\text{Th}_T[\bar{\varphi}]$  be the theory of the class of all generalized reducts  $\mathcal{M} \upharpoonright [\bar{\varphi}]$ , where  $\mathcal{M}$  ranges over models of  $T$ . The proof of the following corollary is just like the proof of Corollary 2.6.

COROLLARY 2.7. *Suppose  $\bar{\varphi} = \varphi_0(\bar{x}), \varphi_1(\bar{x}), \dots, \varphi_k(\bar{x})$  is a tuple of formulas,  $T \supseteq \text{PA}$  is a  $\Sigma_{n+1}^0$  extension of PA, and  $\text{Th}_T[\bar{\varphi}]$  is  $(n+1)$ -thin. Then there is a  $\Delta_{n+1}^0$ -recursively saturated model  $\mathcal{M} \models T$  such that  $\mathcal{M} \upharpoonright [\bar{\varphi}]$  is recursive.*  $\dashv$

**§3. Examples.** We saw in Lemma 2.5 that being  $(n+1)$ -rich and being  $(n+1)$ -thin are incompatible. This section contains examples showing that there are no further restrictions.

THEOREM 3.1. *If  $n < \omega$ , then there is a formula  $\varphi(\bar{x})$  such that the theory  $T_n = \text{Th}[\varphi]$  is  $(n+1)$ -rich,  $n$ -thin, complete and decidable.*

To prove this theorem, we will make use of a construction from [22]. Consider a finite relational language  $\mathcal{L}$ . By consolidating all the relation symbols into just one, we can assume that  $\mathcal{L}$  consists of just one relation symbol  $R$ . We make a further simplification, for expository purposes only, by assuming that  $R$  is a binary relation symbol.

Let  $\mathcal{L}^* = \{U, E, S\}$ , where  $U$  is a new unary relation symbol,  $E$  a new binary relation symbol, and  $S$  is a new 4-ary relation symbol. Given an  $\mathcal{L}$ -structure  $\mathfrak{A}$ , we will define an  $\mathcal{L}^*$ -structure  $\mathfrak{A}^*$ . Then, if  $T$  is an  $\mathcal{L}$ -theory, we let  $T^* = \text{Th}(\{\mathfrak{A}^* : \mathfrak{A} \models T\})$ .

Suppose that  $\mathfrak{A} = (A, R)$ . For each pair  $\langle a, b \rangle \in A^2$ , let  $(Q_{ab}, <_{ab})$  be a countable, dense linearly ordered set such that:

- $(Q_{ab}, <_{ab})$  has a first element iff  $\mathfrak{A} \models R(a, b)$ , and
- $(Q_{ab}, <_{ab})$  has a last element iff  $\mathfrak{A} \models \neg R(a, b)$ .

Thus,  $(Q_{ab}, <_{ab})$  has either a first or last element, but not both. Assume further that the  $Q_{ab}$ 's are pairwise disjoint and that each is disjoint from  $A$ . Now let

$$A^* = A \cup \bigcup \{Q_{ab} : a, b \in A\}.$$

To get the structure  $\mathfrak{A}^* = (A^*, U, E, S)$ , we define the three relations as follows:

- $U = A$ ,
- $E$  is an equivalence relation for which  $A$  and each of  $Q_{ab}$  are equivalence classes,
- if  $a, b, x, y \in A^*$ , then  $\langle a, b, x, y \rangle \in S$  iff  $a, b \in A$ ,  $x, y \in Q_{ab}$  and  $x \leq_{ab} y$ .

[Note: In general, if  $R$  is  $k$ -ary rather than binary, then  $S$  will be  $(k+2)$ -ary.]

The proof of the next lemma is left to the reader. Any number of various standard methods can be used to prove part (1).

LEMMA 3.1.1. (1) *If  $T$  is complete, then  $T^*$  is complete.*

(2) *If  $T$  is decidable, then  $T^*$  is decidable.*

(3) *If  $T = \text{Th}[\bar{\varphi}]$ , then there is a  $\bar{\psi}$  such that  $T^* = \text{Th}[\bar{\psi}]$ .*  $\dashv$

The next lemma is a “stepping-up” lemma for the notion of richness.

LEMMA 3.1.2. *If  $n < \omega$  and  $T$  is  $(n+1)$ -rich, then  $T^*$  is  $(n+2)$ -rich.*

PROOF. Suppose  $\theta(\bar{x})$  is any  $\exists_{n+1}$   $\mathcal{L}$ -formula. Construct the  $\mathcal{L}^*$ -formula  $\theta^*(\bar{x})$  by doing the following. First, replace  $\theta(\bar{x})$  by  $\theta(\bar{x}) \wedge \bigwedge U(x_i)$ , where the conjunction is over all free variables of  $\theta(\bar{x})$ . Second, replace each quantifier  $\exists w$  or  $\forall w$  by  $\exists w \in U$  or  $\forall w \in U$ . Third and last, replace each occurrence of  $R(x, y)$  by either

$$\exists u \forall v [S(x, y, u, u) \wedge (E(u, v) \rightarrow S(x, y, u, v))]$$

or

$$\forall u \exists v [S(x, y, u, u) \rightarrow (u \neq v \wedge S(x, y, u, v))],$$

whichever will result in a  $\exists_{n+2}$  formula. (For example, if  $n$  is even and the occurrence of  $R(x, y)$  is positive, use the first formula.) Roughly, the first formula asserts that  $Q_{xy}$  has a first element, and the second formula that  $Q_{xy}$  does not have a last element. It is routine to check that if  $\bar{a}$  is an appropriate tuple from  $A$ , then

$$\mathfrak{A} \models \theta(\bar{a}) \iff \mathfrak{A}^* \models \theta^*(\bar{a}).$$

Thus, if we let  $\langle \theta_i(\bar{x}) : i < \omega \rangle$  demonstrate that  $T$  is  $(n+1)$ -rich, then  $\langle \theta_i^*(\bar{x}) : i < \omega \rangle$  will demonstrate that  $T^*$  is  $(n+2)$ -rich.  $\dashv$

LEMMA 3.1.3. *Suppose  $m < \omega$  and  $\mathfrak{B}$  is a  $\Delta_{m+2}^0$  model of  $T$ . Then there is  $\mathfrak{A}$  such that  $\mathfrak{A}$  is  $\Delta_{m+1}^0$  and  $\mathfrak{A} \cong \mathfrak{B}^*$ .*

PROOF. We will prove the lemma only in the case that  $m = 0$ . It is easily seen that the proof relativizes to any  $\mathcal{O}^{(m)}$  (or even to any subset of  $\omega$ ).

Thus, let  $\mathfrak{B}$  be a  $\Delta_2^0$  model of  $T$ . Clearly, we can assume that  $B$  is recursive. Now consider  $\mathfrak{B}^* = (B^*, B, E, S)$ . We can also assume that  $B^* = \omega$  and that  $E$  and  $\{\langle a, b, x \rangle \in \omega^3 : \langle a, b, x, x \rangle \in R\}$  are recursive. For each  $\langle a, b \rangle \in B$ , let  $Q_{ab} = \{x \in \omega : \langle a, b, x, x \rangle \in R\}$  and let  $<_{ab}$  be the linear ordering of  $Q_{ab}$ , where  $x <_{ab} y$  iff  $\langle a, b, x, y \rangle \in S$ . Our goal is to construct a recursive  $S' \subseteq \omega^4$  such that there is a permutation  $f$  of  $\omega$  such that  $f : \mathfrak{B}^* \rightarrow (\omega, B, E, S')$  is an isomorphism that is the identity on  $B$ . To get  $S'$  we will uniformly obtain recursive  $<_{ab}'$  that linearly order  $Q_{ab}$  such that  $(Q_{ab}, <_{ab}') \cong (Q_{ab}, <_{ab})$ , and then let  $R'$  be such that  $\langle a, b, x, y \rangle \in R'$  iff  $x <_{ab}' y$ .

Since  $R$  is  $\Delta_2^0$ , we can let  $g : \omega^3 \rightarrow \{0, 1\}$  be such that if  $a, b \in B$ , then  $\langle a, b \rangle \in R$  iff  $\lim_w g(a, b, w) = 1$ . For  $s < \omega$ , let  $Q_{ab}^s$  consist of the first  $2^s$  elements in  $Q_{ab}$ . We define inductively linear orders  $<_{ab}^s$  of  $Q_{ab}^s$  so that  $(Q_{ab}^s, <_{ab}^s) \subseteq (Q_{ab}^{s+1}, <_{ab}^{s+1})$ . Suppose we have  $<_{ab}^s$  so that  $q_0, q_1, q_2, \dots, q_{2^s-1}$  are the elements of  $Q_{ab}^s$  in  $<_{ab}^s$ -increasing order, and let  $p_0, p_1, \dots, p_{2^s-1}$  be the elements of  $Q_{ab}^{s+1} \setminus Q_{ab}^s$  in  $<_{ab}^{s+1}$ -increasing order. Define  $<_{ab}^{s+1}$  on  $Q_{ab}^{s+1}$  so that:

- if  $g(a, b, s) = 0$ , then the elements of  $Q_{ab}^{s+1}$  in  $<_{ab}^{s+1}$ -increasing order are  $p_0, q_0, p_1, q_1, \dots, p_{2^s-1}, q_{2^s-1}$ ,
- if  $g(a, b, s) = 1$ , then the elements of  $Q_{ab}^{s+1}$  in  $<_{ab}^{s+1}$ -increasing order are  $q_0, p_0, q_1, p_1, \dots, q_{2^s-1}, p_{2^s-1}$ .

It should be clear that  $(Q_{ab}, <_{ab}') = \bigcup_s (Q_{ab}^s, <_{ab}^s)$  is as needed.  $\dashv$

We would like a “stepping-up” lemma for the notion of thinness analogous to Lemma 3.1.2. It seems that to get one we need a strengthening of this notion.

DEFINITION 3.1.4. Let  $1 \leq n < \omega$ , and let  $T$  be an  $\mathcal{L}$ -theory. We say that  $T$  is *uniformly  $n$ -thin* if the following hold:

- (1)  $\mathcal{L}$  is a finite language, and  $T$  is a recursively axiomatizable theory;
- (2) whenever  $m < \omega$ ,  $\mathcal{L}' \supseteq \mathcal{L}$  is a finite language and  $T' \supseteq T$  is a  $\Sigma_{n+m}^0$   $\mathcal{L}'$ -theory, then  $T'$  has a model  $\mathfrak{A}'$  such that  $\mathfrak{A} = \mathfrak{A}' \upharpoonright \mathcal{L}$  is  $\Delta_{1+m}^0$ .

We say that  $T$  is *uniformly 0-thin* if (1) holds.

LEMMA 3.1.5. *If  $n < \omega$  and  $T$  is uniformly  $n$ -thin, then  $T^*$  is uniformly  $(n+1)$ -thin.*

PROOF. Suppose that  $m < \omega$ , that  $\mathcal{L}' \supseteq \mathcal{L}^*$  is a finite language, and that  $T' \supseteq T^*$  is a  $\Sigma_{n+1+m}^0$   $\mathcal{L}'$ -theory. Our aim is to show that  $T'$  has a model  $\mathfrak{A}'$  such that  $\mathfrak{A}' \restriction \mathcal{L}^*$  is  $\Delta_{1+m}^0$ .

Given a countable model  $\mathfrak{A} \models T^*$ , we can get in a natural way an  $\mathcal{L}$ -model  $\mathfrak{A}^- \models T$  that is definable in  $\mathfrak{A}$  such that  $(\mathfrak{A}^-)^* \cong \mathfrak{A}$ .

Since  $T$  is uniformly  $n$ -thin, we can let  $\mathfrak{B}$  be a model of  $T'$  such that  $(\mathfrak{B} \restriction \mathcal{L}^*)^-$  is  $\Delta_{2+m}^0$ . (Notice that  $n = 0$  causes no problem.) By Lemma 3.1.3, there is  $\Delta_{1+m}^0$   $\mathfrak{A} \models T^*$  such that  $\mathfrak{A} \cong \mathfrak{B} \restriction \mathcal{L}^*$ . Let  $\mathfrak{A}'$  be an expansion  $\mathfrak{A}$  such that  $\mathfrak{A}' \cong \mathfrak{B}$ . Clearly,  $\mathfrak{A}'$  is as required.  $\dashv$

PROOF OF THEOREM 3.1. Let  $T_0 = \text{Pr}$ , and then inductively let  $T_{n+1} = T_n^*$ . We know that  $T_0$  is 1-rich and uniformly 0-thin and that  $T_0 = \text{Pr} = \text{Th}[x = y + z]$ , which is a complete, decidable theory. By Lemmas 3.1.2, 3.1.5 and 3.1.1, each  $T_n$  is  $(n+1)$ -rich, uniformly  $n$ -thin, complete and decidable, and  $T_n = \text{Th}[\varphi]$  for some formula  $\varphi(\bar{x})$ .  $\dashv$

COROLLARY 3.2. *If  $n < \omega$ , then there is a formula  $\varphi(\bar{x})$  such that:*

- (1) *if  $\mathcal{M} \models \text{PA}$  is nonstandard and  $\varnothing^{(n+1)} \in \text{SSy}(\mathcal{M})$ , then  $\mathcal{M} \restriction [\varphi]$  is not recursive;*
- (2) *there is a recursively saturated,  $n$ -correct model  $\mathcal{M} \models \text{PA}$  such that  $\mathcal{M} \restriction [\varphi]$  is recursive.*

PROOF. This follows from Theorems 3.1, 1.2 and Corollary 2.6.  $\dashv$

Let  $\mathcal{L}_{\text{LO}} = \{<\}$  be the language appropriate for linearly ordered sets. Consider the ordinal  $\omega^\omega$ , and then let  $T_\omega = \text{Th}((\omega^\omega, <))$ . It is known (see Theorem 13.46 of [33]) that  $T_\omega$  is a decidable  $\mathcal{L}_{\text{LO}}$ -theory and also that  $T_\omega$  is the theory of the linearly ordered “set” consisting of all ordinals. Now let  $\varphi_\omega(x, y)$  be a natural  $\Delta_1^0$  formula  $\varphi_\omega(x, y)$  that defines, in the standard model, a linear order of order type  $\omega^\omega$ . [For example, let  $\varphi_\omega(x, y)$  be the formula asserting: If  $X, Y$  are the “finite” sets canonically coded by  $x, y$  respectively, then either  $|X| < |Y|$  or else  $|X| = |Y|$  and there is  $a \in X \setminus Y$  such that  $[0, a) \cap X = [0, a) \cap Y$ .] Then PA will prove that this formula defines an ordering of order type  $\omega^\omega$  in the sense that all the sentences used in defining  $\omega^\omega$  by recursion are consequences of PA. This formula is unique in the sense that any other formula with the same property defines an ordering that is definably isomorphic to the one defined by  $\varphi_\omega(x, y)$ . It can be shown that  $T_\omega = \text{Th}[\varphi_\omega]$ .

THEOREM 3.3. *The theory  $T_\omega$  is complete, decidable, rich and  $n$ -thin for each  $n < \omega$ .*

PROOF. That  $T_\omega$  is complete and decidable has already been commented on.

We show that  $T_\omega$  is rich. Let  $\theta_0(x)$  be the formula  $x = x$ . Inductively, let  $\theta_{i+1}(x)$  assert that the largest  $a \leq x$  for which  $\theta_i(a)$  is a limit of points  $b$  such

that  $\theta_i(b)$ . In the case of “the standard model”  $(\omega^\omega, <)$  a typical element  $\alpha$  (that is,  $\alpha < \omega^\omega$ ) is such that  $\alpha = \omega^n c_n + \omega^{n-1} c_{n-1} + \cdots + \omega c_1 + c_0$ , where each  $c_i < \omega$  and if  $n > 0$ , then  $c_n > 0$ . Then,  $\alpha$  satisfies  $\theta_{i+1}(x)$  iff  $c_i = 0$  and  $n \geq i + 1$ . Clearly, this sequence shows that  $T$  is rich.

What remains to be proved is that  $T_\omega$  is  $n$ -thin for each  $n < \omega$ . We will actually prove that  $T_\omega$  is uniformly  $n$ -thin for each  $n < \omega$ . But before doing so, we make a little digression to discuss a construction analogous to the one used in proving Theorem 3.1.

We review some well known properties of linearly ordered sets. An excellent reference is [33]. It is well known that  $\text{Th}((\omega, <))$  is finitely axiomatizable and, therefore, decidable. If  $(A, <)$  is any linearly ordered set, then  $(A, <) \equiv (\omega, <)$  iff there is a possibly empty order type  $\rho$  such that  $(A, <)$  has order type  $\omega + (\omega^* + \omega) \cdot \rho$ . If  $(A, <) \equiv (\omega, <)$  is countable and recursively saturated, then  $(A, <)$  has order type  $\omega + (\omega^* + \omega) \cdot \eta$ , which is the order type of every countable nonstandard model of PA. (Recall that  $\eta$  is the order type of the rationals.)

If  $(A, <)$  has order type  $\rho$  and  $(B, <)$  is obtained from  $(A, <)$  by replacing each point of  $A$  with a copy of  $\omega$ , then  $(B, <)$  has order type  $\omega \cdot \rho$ . If  $(C, <)$  is obtained from  $(A, <)$  by replacing each point  $a \in A$  with a set having order type  $\omega + (\omega^* + \omega) \cdot \rho_a$  (where  $\rho_a$  is some order type depending on  $a$ ), then  $(C, <) \equiv (B, <)$ . In particular, if  $(B, <)$  and  $(C, <)$  have order types  $\omega \cdot \alpha$  and  $(\omega + (\omega^* + \omega) \cdot \eta) \cdot \alpha$  respectively, then  $(B, <) \equiv (C, <)$ .

The following lemma is easily proved.

**LEMMA 3.3.1.** *Suppose that  $(A, <)$  is a  $\Delta_2^0$  linearly ordered set having order type  $\alpha$ . Then there is a recursive  $(B, <)$  having order type  $(\omega + (\omega^* + \omega) \cdot \eta) \cdot \alpha$ .*

Observe that this lemma relativizes: if  $X \subseteq \omega$  and  $(A, <)$  is recursive in  $X'$  (instead of being  $\Delta_2^0$ ), then we get  $(B, <)$  that is recursive in  $X$  (instead of being recursive). In particular, if we suppose that  $(A, <)$  is  $\Delta_{n+2}^0$ , then we can get  $(B, <)$  to be  $\Delta_{n+1}^0$ . This lemma is undoubtedly not optimal; see the closely related discussion in [4, §5].

If  $T$  is any theory of linearly ordered sets, then let  $T^*$  be the theory of all linearly order sets  $(A^*, <)$  such that there is  $(A, <) \models T$  having order type  $\alpha$  and  $(A^*, <)$  has order type  $\omega \cdot \alpha$ .

**LEMMA 3.3.2.** *If  $T$  is a uniformly  $n$ -thin theory of linearly ordered sets, then  $T^*$  is uniformly  $(n + 1)$ -thin.*

**PROOF.** Suppose  $T$  is a uniformly  $n$ -thin  $\mathcal{L}_{\text{LO}}$ -theory all of whose models are linearly ordered sets. Let  $\mathcal{L}_0 \supseteq \mathcal{L}_{\text{LO}}$  be finite, and let  $T_0^* \supseteq T^*$  be a  $\Sigma_{n+m}^0 \mathcal{L}_0$ -theory, where  $n + m \geq 1$ . Our goal is to show that there is a model  $\mathfrak{A}_0^* \models T_0^*$  such that  $\mathfrak{A}_0^* \upharpoonright \mathcal{L}_{\text{LO}}$  is  $\Delta_{1+m+1}^0$ .

Let  $\mathcal{L}_1 = \mathcal{L}_0 \cup \{U, S, +, \cdot\}$ , where  $U$  is a unary relation symbol,  $+$  and  $\cdot$  are binary function symbols, and  $S$  is a unary function symbol. Get the theory  $T_1^*$  by adding to  $T_0^*$  sentences asserting:

- $S$  is the successor function;
- $U$  is the set of limit points;
- if  $x, y \in U$  and  $[x, y] = \{z : x \leq z < y\}$  is such that  $[x, y] \cap U = \{x\}$ , then  $([x, y], +, \cdot, \leq, x, S(x))$  is a model of PA.

Clearly,  $T_1^*$  is a  $\Sigma_{n+m}^0$ -theory, so it has a recursively saturated model  $\mathfrak{A}_1^*$  such that  $(\mathfrak{A}_1^* \upharpoonright U) \upharpoonright \mathcal{L}_{LO}$  is  $\Delta_{1+m}^0$ . Let  $\alpha$  be the order type of  $(\mathfrak{A}_1^* \upharpoonright U) \upharpoonright \mathcal{L}_{LO}$ . Each of the models of PA that  $T_1^*$  claims to exist has order  $\omega + (\omega^* + \omega) \cdot \eta$ , so the order type of  $\mathfrak{A}_0^* \upharpoonright \mathcal{L}_{LO}$  is  $(\omega + (\omega^* + \omega) \cdot \eta) \cdot \alpha$ . By the relativized version of Lemma 3.3.1, we can arrange that  $\mathfrak{A}_0^* \upharpoonright \mathcal{L}_{LO}$  is  $\Delta_{1+m+1}^0$ .  $\dashv$

*Completing the proof of Theorem 3.3.* First,  $T_\omega$  is uniformly 0-thin. Clearly,  $\omega \cdot \omega^\omega = \omega^{1+\omega} = \omega^\omega$ , so that  $T_\omega = (T_\omega)^*$ . Thus, repeated applications of Lemma 3.3.2 yield that  $T_\omega$  is uniformly  $n$ -thin for each  $n < \omega$ .  $\dashv$

**COROLLARY 3.4.** *Let  $\varphi(x, y)$  be a formula such that  $T_\omega = \text{Th}[\varphi]$ . Then:*

- (1) *if  $\mathcal{M} \models \text{TA}$  and  $\mathcal{M}$  is recursively saturated, then  $\mathcal{M} \upharpoonright [\varphi]$  is not arithmetic;*
- (2) *for each  $n < \omega$ , there is a recursively saturated,  $n$ -correct model  $\mathcal{M} \models \text{PA}$  such that  $\mathcal{M} \upharpoonright [\varphi]$  is recursive.*

**PROOF.** This follows from Theorems 3.3, 1.2 and Corollary 2.6.  $\dashv$

Corollary 3.4 leaves unanswered the following question concerning the formula  $\varphi_\omega(x, y)$ .

**QUESTION 3.5.** Is there a nonstandard model  $\mathcal{M} \models \text{TA}$  such that  $\mathcal{M} \upharpoonright [\varphi_\omega]$  is recursive?

By a *tree* we mean a partially ordered set  $(A, <)$  in which the set of predecessors of any element is linearly ordered. In Corollary 4.2 in the next section, we will see that the theory of trees is 1-thin. The next proposition implies that the theory of trees is not 2-thin. Incidentally, an example was given in [22] of a complete  $\Delta_2^0$  theory of trees that has no recursive model.

**PROPOSITION 3.6.** *There is a formula  $\varphi(x, y)$  such that:*

- (a)  *$\text{Th}[\varphi]$  is a complete, decidable 2-rich theory of trees;*
- (b) *if  $\mathcal{M}$  is nonstandard and  $\emptyset' \in \text{SSy}(\mathcal{M})$ , then  $\mathcal{M} \upharpoonright [\varphi]$  is not recursive.*

**PROOF.** Clearly, Theorem 1.2 implies that (b) follows from (a).

To prove (a), we first define a complete, decidable 2-rich theory  $T$  of trees. For each  $i < \omega$ , let  $\theta_i(x)$  be the  $\exists_2$  formula

$$\exists x_0, x_1, \dots, x_i \left[ x < x_0 < x_1 < \dots < x_i \wedge \right.$$

$$\left. \forall y \left( (y > x_0 \vee x < y \leq x_0) \rightarrow y \in \{x_0, x_1, \dots, x_i\} \right) \right].$$

Let  $T'$  be the theory of trees consisting of all sentences

$$\exists x \left[ \bigwedge_{i \in I} \theta_i(x) \wedge \bigwedge_{i \in J} \neg \theta_i(x) \right],$$

where  $I, J \subseteq \omega$  are disjoint finite sets. Clearly,  $T'$  is a 2-rich theory of trees.

Now let  $T$  be the theory of trees consisting of the recursive set of sentences asserting:

- (1) there is a unique root;
- (2) the root has infinitely many immediate successors (called *atoms*);
- (3) each atom has infinitely many immediate successors (called *btoms*);
- (4) for any  $x$  that is not an atom, a btom or the root, there are an atom  $a$  and a btom  $b$  such that  $a < b < x$ ;
- (5) for any btom  $b$ , the set  $\{x : x \geq b\}$  is linearly ordered with first and last elements such that every element but the first has an immediate predecessor and every element but the last has an immediate successor.
- (6) for each  $n < \omega$ , if  $a$  is an atom and  $b, b' > a$  are distinct btoms such that  $|\{x : x > b\}| = n$ , then  $|\{x : x > b'\}| \neq n$ .
- (7) if  $I, J \subseteq \omega$  are disjoint finite sets, then there is an atom  $a$  such that if  $N = \{x : x > b : b > a \text{ is a btom}\}$ , then  $I \subseteq N$  and  $J \cap N = \emptyset$ .

Clearly,  $T$  is a consistent theory of trees. Various standard ways can be used to show that  $T$  is complete, so it is decidable. Also, from (7) it is clear that  $T \vdash T'$ , so that  $T$  is 2-rich.

It is straightforward to come up with  $\varphi(x, y)$  that encodes all of  $T$ , and then showing that  $T = \text{Th}[\varphi]$ . ⊢

We will see in Corollary 5.2 that the theory of linearly ordered sets is 2-thin. The next proposition shows that this theory is not 3-thin. Incidentally, an example was given in [22] of a complete  $\Delta_3^0$  theory of linearly ordered sets that has no recursive model.

**PROPOSITION 3.7.** *There is a formula  $\varphi(x, y)$  such that:*

- (a)  $\text{Th}[\varphi]$  is a complete, decidable 3-rich theory of linearly ordered sets;
- (b) if  $\mathcal{M}$  is nonstandard and  $\emptyset^{(2)} \in \text{SSy}(\mathcal{M})$ , then  $\mathcal{M} \models [\varphi]$  is not recursive.

**PROOF.** For  $i < \omega$ , let  $\psi_i(u, v)$  be the  $\exists_3$  formula asserting that  $u < v$ ,  $u$  has no immediate predecessor,  $v$  has no immediate successor, and there are exactly  $i$  elements between  $u$  and  $v$ . Let  $\theta_i(x, y)$  be the  $\exists_3$  formula

$$\exists uv [x < u < v < y \wedge \psi_i(u, v)].$$

Using these formulas, we can define the 3-rich theory  $T'$  much as in the proof of Proposition 3.6. It is left to the reader to complete this proof in the manner of the proof of Proposition 3.6. ⊢

**§4. Some 1-thin theories.** In this section, we exhibit some 1-thin theories.

We will be considering classes of structures in this section. Whenever we refer to a class  $K$  of structures, it is to be understood that there is some language  $\mathcal{L}$  (that, according to our conventions, contains no function symbols and, unless specified otherwise, is finite) such that  $K$  is a class of  $\mathcal{L}$ -structures closed under isomorphism. We say that the class  $K$  is *hereditary* if whenever  $\mathfrak{B} \subseteq \mathfrak{A} \in K$ , then  $\mathfrak{B} \in K$ .

Now suppose  $K$  is a class of finite structures. We let  $K_\infty$  be the class of structures  $\mathfrak{A}$  such that if  $X \subseteq A$  is finite, then there is  $\mathfrak{B} \in K$  such that  $X \subseteq B$  and  $\mathfrak{B} \subseteq \mathfrak{A}$ . Clearly,  $K$  is the class of those finite structures in  $K_\infty$ . We let  $T_K = \text{Th}(K_\infty)$ . If  $K$  is hereditary, then  $\forall \cap \text{Th}(K)$  is an axiomatization for  $T_K$ , and  $K_\infty$  is the class of models of  $T_K$ . For example, if  $K$  is the class of finite trees, then  $K$  is hereditary,  $K_\infty$  is the class of all trees, and  $T_K$  is the theory of trees.

A class  $K$  is *well-quasi-ordered* (abbreviation: *wqo*) if whenever  $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$  is an infinite sequence of structures in  $K$ , then there are  $i < j < \omega$  such that  $\mathfrak{A}_i$  is embeddable in  $\mathfrak{A}_j$ . We will be interested in wqo only for those classes  $K$  consisting of finite structures. A class  $K$  of finite structures is wqo iff whenever  $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$  is an infinite sequence of structures in  $K$ , then there are distinct  $i, j < \omega$  such that  $\mathfrak{A}_i$  is embeddable in  $\mathfrak{A}_j$ .

The notion of wqo has wider applicability. First,  $(Q, \sqsubseteq)$  is a quasi-order if  $\sqsubseteq$  is a reflexive and transitive relation on  $Q$ . Thus, any class  $K$  of structures is quasi-ordered by embeddability. A quasi-order  $(Q, \sqsubseteq)$  is wqo iff whenever  $a_0, a_1, a_2, \dots, a_i, \dots \in Q$ , then there are  $i < j < \omega$  such that  $a_i \sqsubseteq a_j$ . A fundamental theorem of Higman [9] is that if  $(Q, \sqsubseteq)$  is wqo, then so is  $Q^{<\omega}$ , the set of finite sequences from  $Q$  ordered by:  $(a_0, a_1, \dots, a_{m-1}) \sqsubseteq (b_0, b_1, \dots, b_{n-1})$  iff there are  $0 \leq k_0 < k_1 < \dots < k_{m-1} < n$  such that  $a_i \sqsubseteq b_{k_i}$  for all  $i < m$ .

A class of finite structures that is obviously wqo is the class  $\mathcal{S}$  of all structures  $(A, <, S)$  such that for some  $k < \omega$ ,  $A = \{a_0, a_1, \dots, a_{k-1}\}$  and  $<, S$  are binary relations on  $A$  such that  $a_i < a_j \iff i < j$  and  $a_i S a_j \implies j = i + 1$ . It follows by Higman's Theorem that the class of all substructures of structures in  $\mathcal{S}$ , is also wqo.

If  $K$  is a class and  $n < \omega$ , then  $K^{(n)}$  is the class consisting of all expansions  $(\mathfrak{A}, a_0, a_1, \dots, a_{n-1})$ , where  $\mathfrak{A} \in K$  and  $a_0, a_1, \dots, a_{n-1} \in A$ . We say that  $K$  is *strongly wqo* if  $K^{(n)}$  is wqo for each  $n < \omega$ . The class  $\mathcal{S}$  (from the previous paragraph), while being wqo, is not strongly wqo. In fact,  $\mathcal{S}^{(2)}$  is not wqo as can be seen by considering  $(\mathfrak{A}_n, c_0, c_1)$ , where  $A_n = \{a_0, a_1, \dots, a_n\}$  such that  $c_0 = a_0 S a_1 S a_2 \dots a_{n-1} S a_n = c_1$ .

There is a notion stronger than strongly wqo. We say that the class  $K$  is *very wqo* if, for each  $n < \omega$ , the class of structures  $(\mathfrak{A}, U_0, U_1, \dots, U_{n-1})$ , where  $\mathfrak{A} \in K$  and  $U_0, U_1, \dots, U_{n-1} \subseteq A$ , is wqo. Clearly, if  $K$  is very wqo, then it is strongly wqo.

If  $\mathbf{K}$  is a class of finite structures and we say that  $\mathbf{K}$  is finite, recursive, etc., then we mean that the set (of canonical codes) of the isomorphism types of structures in  $\mathbf{K}$  is finite, recursive, etc. Clearly, if  $\mathbf{K}$  is a recursive, hereditary class of finite structures, then  $T_{\mathbf{K}}$  is recursively axiomatizable. A theorem of Pouzet [32] says that if  $\mathbf{K}$  is a very wqo class of finite structures, then  $T_{\mathbf{K}}$  is finitely axiomatizable. (More about Pouzet's theorem appears right after Lemma 4.7.) The class of cycle-free finite graphs shows that this is not always the case.

We introduce some terminology and notation that will be useful in the proof of Theorem 4.1 and elsewhere in this and the next section. Suppose we are considering a class  $\mathbf{K}$  of finite  $\mathcal{L}$ -structures. Let  $c_0, c_1, c_2, \dots$  be an infinite supply of new and distinct constant symbols to be used in a Henkin construction. If  $N \subseteq \omega$ , then let  $\mathcal{L}_N = \mathcal{L} \cup \{c_n : n \in N\}$ . In particular, if  $s < \omega$ , then  $\mathcal{L}_s = \mathcal{L} \cup \{c_0, c_1, \dots, c_{s-1}\}$ , and also  $\mathcal{L}_\omega = \bigcup_s \mathcal{L}_s$ . Recall from §0 that  $\nabla$  is the set of literals. If  $s < \omega$ , then by a  $(\mathbf{K}, \mathcal{L}_s)$ -*diagram* we will mean a (necessarily finite) set  $D \subseteq \nabla$  of  $\mathcal{L}_s$ -sentences such that: (1) if  $\alpha$  is an atomic  $\mathcal{L}_s$ -sentence, then either  $\alpha \in D$  or  $\neg\alpha \in D$ ; (2) the sentence  $c_i = c_j$  is in  $D$  iff  $i = j < s$ ; (3) if  $c \in \mathcal{L}$  is a constant symbol, then the sentence  $c_i = c$  is in  $D$  for some  $i < s$ ; and (4) the  $\mathcal{L}_s$ -structure whose diagram is  $D$  is an expansion of a structure in  $\mathbf{K}$ .

The following theorem is essentially that of Ershov [6] but with a slightly stronger conclusion.

**THEOREM 4.1.** *Suppose that  $\mathbf{K}$  is a recursive, strongly wqo, hereditary class of finite structures. Then  $T_{\mathbf{K}}$  is 1-thin.*

**PROOF.** Let  $\mathcal{L}$  be the language for  $\mathbf{K}$  and assume, without loss of generality, that there are no constant symbols in  $\mathcal{L}$ . As already noted,  $T_{\mathbf{K}}$  is recursively axiomatized by  $\forall \cap \text{Th}(\mathbf{K})$ . Let  $\mathcal{L}' \supseteq \mathcal{L}$  be a finite language, and let  $T' \supseteq T_{\mathbf{K}}$  be a recursively axiomatizable  $\mathcal{L}'$ -theory. Our goal is to construct a model  $\mathfrak{A}' \models T'$  such that  $\mathfrak{A} = \mathfrak{A}' \upharpoonright \mathcal{L}$  is recursive. Clearly, we can let  $\mathfrak{A}'$  be a finite model of  $T'$  if there is one; therefore, we assume that  $T'$  has no finite models.

Let  $\mathbf{K}'$  be the subclass of  $\mathbf{K}$  consisting of all  $\mathfrak{A} \in \mathbf{K}$  that are substructures of  $\mathcal{L}$ -reducts of models of  $T'$ . Clearly,  $\mathbf{K}$  is strongly wqo and hereditary. It is also recursive. For, let  $\mathbf{F}$  be the class of those  $\mathfrak{A} \in \mathbf{K} \setminus \mathbf{K}'$  that are minimal in the sense that if  $\mathfrak{B}$  is embeddable in  $\mathfrak{A}$  and is not isomorphic to  $\mathfrak{A}$ , then  $\mathfrak{B} \in \mathbf{K}'$ . Since  $\mathbf{K}$  is wqo,  $\mathbf{F}$  is finite. Then, since  $\mathbf{K}'$  consists of those structures in  $\mathbf{K}$  having no substructures in  $\mathbf{F}$ ,  $\mathbf{K}'$  is recursive.

We have just seen that  $\mathbf{K}'$  is a recursive, strongly wqo, hereditary class of finite structures, so without loss of generality, we will assume that  $\mathbf{K} = \mathbf{K}'$ .

The model  $\mathfrak{A}$  will be constructed by a Henkin construction using new constant symbols  $c_0, c_1, c_2, \dots$ . Let  $\mathcal{L}'_s = \mathcal{L}' \cup \{c_0, c_1, \dots, c_{s-1}\}$  and  $\mathcal{L}'_\omega = \bigcup_s \mathcal{L}'_s$ . Our goal is to construct a complete, Henkin  $\mathcal{L}'_\omega$ -theory  $T'' \supseteq T'$  such that  $T'' \cap \nabla$  is recursive.

*A short excursus:* Let  $T''$  be a set of  $\mathcal{L}'_\omega$ -sentences. The reader is reminded that  $T''$  is Henkin iff for every  $\mathcal{L}'_\omega$ -formula  $\varphi(x)$ , there is  $i < \omega$  such that  $c_i$  is a Henkin witness for  $\varphi(x)$ , that is, the sentence  $\exists x\varphi(x) \rightarrow \varphi(c_i)$  is in  $T''$ . If  $T''$  is a complete Henkin theory, then its Henkin model is a model of  $T''$ . Notice that if  $T''$  is a Henkin theory and each sentence  $c_i \neq c_0$  is in  $T$  for  $i > 0$ , then  $T''$  is complete. For, consider an  $\mathcal{L}'_\omega$ -sentence  $\sigma$ , and then let  $\varphi(x)$  be the formula  $\sigma \leftrightarrow x = c_0$ . Since  $T''$  is Henkin, let  $c_i$  be a Henkin witness for  $\varphi(x)$ . Then  $T'' \vdash \sigma$  iff  $i = 0$  and  $T'' \vdash \neg\sigma$  iff  $i \neq 0$ . We will be constructing a Henkin theory  $T''$  such that  $\{c_i \neq c_j : i < j < \omega\} \subseteq T''$ , so that  $T''$  will necessarily be complete.

Returning to the proof, let  $\sigma_0, \sigma_1, \sigma_2, \dots$  be a recursive list of all  $\mathcal{L}'_\omega$ -sentences that can be logically deduced from  $T'$ , where each  $\sigma_i$  is an  $\mathcal{L}'_i$ -sentence. If  $\Gamma$  is a set of  $\mathcal{L}'_\omega$ -sentences, then we say that  $\Gamma$  is *s-consistent* if no contradiction can be derived from  $\Gamma \cup \{\sigma_0, \sigma_1, \dots, \sigma_{s-1}\}$  in  $s$  steps.

Since  $\mathbf{K}$  is recursive, it follows that the set

$$\mathcal{E} = \{D : D \text{ is a } (\mathbf{K}, \mathcal{L}_s)\text{-diagram for some } s < \omega\}$$

is recursive. If  $D = \bigcup_i D_{r_i}$ , where  $r_0 < r_1 < r_2 < \dots < \omega$ ,  $D_{r_0} \subseteq D_{r_1} \subseteq D_{r_2} \subseteq \dots$  and each  $D_{r_i}$  is a  $(\mathbf{K}, \mathcal{L}_{r_i})$ -diagram, then we say that  $D$  is a  $(\mathbf{K}, \mathcal{L}_\omega)$ -diagram.

Let  $\varphi_0(x), \varphi_1(x), \varphi_2(x), \dots$  be a recursive list of all  $\mathcal{L}'_\omega$ -formulas having  $x$  as the only free variable such that each  $\varphi_i(x)$  is an  $\mathcal{L}'_i$ -formula. For convenience, let  $\varphi_0(x)$  be the formula  $x = x$ . Given  $I \subseteq \omega$  and  $h : I \rightarrow \omega$ , we let

$$H(h) = \{\exists x\varphi_i(x) \rightarrow \varphi_i(c_{h(i)}) : i \in I\}.$$

With this notation, our goal is to get a function  $h : \omega \rightarrow \omega$  and a recursive  $(\mathbf{K}, \mathcal{L}_\omega)$ -diagram  $D$  such that if  $T'' = D \cup H(h) \cup T'$ , then  $T''$  is consistent.

We will construct  $h$  and  $D$  by constructing finite approximations to them. At stage  $s$ , we will define  $N_s \leq s$ , a function  $h_s : N_s \rightarrow s$ , and a  $(\mathbf{K}, \mathcal{L}_s)$ -diagram  $D_s$  in such a way that, for each  $s < \omega$ , the following hold:

- (0)  $D_s \cup T'$  is consistent;
- (1)  $D_s \cup H(h_s)$  is  $s$ -consistent.

We begin by letting  $N_0 = 0$  and  $h_0 = D_0 = \emptyset$ .

At the onset of stage  $s + 1$ , we have a function  $h_s : N_s \rightarrow s$  and a  $(\mathbf{K}, \mathcal{L}_s)$ -diagram  $D_s$  such that (0) and (1) hold. Let  $t \leq N_s$  be the largest for which there are an  $\mathcal{L}_{s+1}$ -diagram  $D_{s+1} \supseteq D_s$  and a function  $h_{s+1} : t + 1 \rightarrow s + 1$  such that  $D_{s+1} \cup T'$  is consistent,  $h_{s+1} \upharpoonright t = h_s \upharpoonright t$  and  $D_{s+1} \cup H(h_{s+1})$  is  $(s + 1)$ -consistent.

Observe that  $t$  is well defined. For, since (0) holds and  $T'$  has no finite models, we can let  $D_{s+1} \supseteq D_s$  be any  $(\mathbf{K}, \mathcal{L}_{s+1})$ -diagram. Then, letting  $t = 0$  and  $h_{s+1} = \{\langle 0, 0 \rangle\}$ , we have that  $D_{s+1} \cup \{\exists x\varphi_0(x) \rightarrow \varphi_0(c_0)\} \cup T'$  is consistent and, therefore,  $D_{s+1} \cup H(h_{s+1})$  is  $(s + 1)$ -consistent.

Having  $t$ , let  $N_{s+1} = t + 1$ , and then choose  $h_{s+1}$  and  $D_{s+1}$  in some effective manner. For each  $s < \omega$ , the following hold:

- (2)  $D_s \subseteq D_{s+1}$ ;
- (3)  $1 \leq N_{s+1} \leq N_s + 1$  and  $h_{s+1} \upharpoonright (N_{s+1} - 1) \subseteq h_s$ .

The crucial fact needed to complete the proof is the following claim.

CLAIM.  $\lim_s N_s = \infty$ .

For, having this claim, we let  $h = \lim_s h_s$ , which by (3), is now well defined and  $h : \omega \rightarrow \omega$ . Let  $D = \bigcup_s D_s$  and  $T'' = D \cup H(h) \cup T'$ . It follows from (1) that  $T''$  is  $s$ -consistent for all  $s < \omega$  and, therefore,  $T''$  is consistent. Clearly,  $D$  is recursive by (2).

It remains to prove the claim. Suppose the claim is false. By (3),  $N_s > 0$  if  $s > 0$ , so we can let  $m = \liminf_s N_s - 1$ . Let  $r < \omega$  be such that  $N_r = m + 1$  and  $N_s \geq m + 1$  for all  $s \geq r$ . By (3),  $h_s \upharpoonright m = h_r \upharpoonright m$  for all  $s \geq r$ . Then let  $r < s_0 < s_1 < s_2 < \dots$  be such that  $N_{s_i} = m + 1$  for all  $i < \omega$ . Let  $\mathfrak{A}_i$  be the  $\mathcal{L}$ -structure whose diagram is  $D_{s_i-1}$ . Let  $a_k = c_{h_r(k)}$  for  $k < m$ , and  $b_i = c_{h_{s_i-1}(m)}$ . Then, let  $\mathfrak{A}_i^* = (\mathfrak{A}_i, a_0, a_1, \dots, a_{m-1}, b_i) \in \mathbf{K}^{(m+1)}$ .

Since  $\mathbf{K}^{(m+1)}$  is wqo, there are  $i < j$  such that  $\mathfrak{A}_i^*$  is embeddable in  $\mathfrak{A}_j^*$ . Moreover, there are  $i_0 < i_1 < i_2 < \dots$  such that each  $\mathfrak{A}_{i_k}^*$  is embeddable in  $\mathfrak{A}_{i_{k+1}}^*$ . We then might as well assume that  $\mathfrak{A}_i^*$  is embeddable in  $\mathfrak{A}_{i+1}^*$ . By composing these embeddings, we get embeddings  $e_i : \mathfrak{A}_0^* \rightarrow \mathfrak{A}_i^*$  for each  $i < \omega$ . Hence,  $D_{s_0-1} \cup H(h_{s_0-1} \upharpoonright (m+1))$  is  $(s_i - 1)$ -consistent for each  $i < \omega$ , and, therefore,  $D_{s_0-1} \cup H(h_{s_0-1} \upharpoonright (m+1)) \cup T'$  is consistent. But then, there must be an  $\mathcal{L}_{s_0}$ -diagram  $D' \supseteq D_0$  and  $h' : m + 2 \rightarrow s_0$  such that  $h' \upharpoonright (m+1) = h_{s_0-1} \upharpoonright (m+1)$  and  $D' \cup H(h') \cup T'$  is consistent. Therefore,  $N_{s_0} \geq m + 2$ , a contradiction that proves the claim and the theorem.  $\dashv$

REMARK. In the previous proof, the model  $\mathfrak{A}'$  we constructed has the additional feature that  $\mathcal{D}(\mathfrak{A}')$  is  $\Delta_2^0$ .

A famous example of a class  $\mathbf{K}$  satisfying the conditions of Theorem 4.1 is the class of finite trees. It is trivial that  $\mathbf{K}$  is recursive and hereditary. That  $\mathbf{K}$  is strongly (even very) wqo is a standard consequence of Kruskal's Theorem [19]. (See Theorem 6.2 for a still stronger result.) For this class  $\mathbf{K}$ ,  $T_{\mathbf{K}}$  is the theory of trees.

COROLLARY 4.2. *The theory of trees is 1-thin.*  $\dashv$

Thus, it follows that every recursively axiomatizable theory of trees has a recursive, recursively saturated model. This slightly strengthens Ershov's theorem [6] for trees by adding the requirement that the model should be recursively saturated. We can now conclude: if  $\varphi(x, y)$  is a formula such that  $\mathcal{M} \upharpoonright [\varphi]$  is a tree for every model  $\mathcal{M} \models \text{PA}$ , then  $\text{Th}[\varphi]$  is 1-thin and then, by Corollary 2.6, there is a recursively saturated model  $\mathcal{M} \models \text{PA}$  for

which  $\mathcal{M} \upharpoonright [\varphi]$  is a recursive tree. This can be improved using Corollary 2.7 by requiring that there be just *some* model  $\mathcal{M}$  for which  $\mathcal{M} \upharpoonright [\varphi]$  is a tree.

**COROLLARY 4.3.** *Suppose  $\varphi(x, y)$  is a formula such that there is a model  $\mathcal{M} \models \text{PA}$  for which  $\mathcal{M} \upharpoonright [\varphi]$  is a tree. Then there is a recursively saturated model  $\mathcal{M} \models \text{PA}$  for which  $\mathcal{M} \upharpoonright [\varphi]$  is a recursive tree.*

**PROOF.** Let  $T$  be the theory  $\text{PA} + \text{"}\varphi(x, y) \text{ defines a tree"}$ . Then  $T$  is a consistent, recursively axiomatizable extension of  $\text{PA}$  and  $\text{Th}_T[\bar{\varphi}]$  is 1-thin. By Corollary 2.7 there is a recursively saturated model  $\mathcal{M} \models T$  for which  $\mathcal{M} \upharpoonright [\varphi]$  is recursive. Since  $\mathcal{M} \models T$ , then  $\mathcal{M} \upharpoonright [\varphi]$  is a tree.  $\dashv$

A consequence of Corollary 4.2 is that, not only is the theory of trees 1-thin, but every recursively axiomatizable theory of trees is 1-thin. Thus, by Lemma 2.5, no recursively axiomatizable theory of trees is 1-rich. But then, no theory of trees is 1-rich. For, suppose that  $T$  is a 1-rich theory of trees and that  $\langle \theta_i(\bar{x}) : i < \omega \rangle$  demonstrates the 1-richness of  $T$ . Then, the tree axioms together with all sentences,

$$\exists \bar{x} \left[ \bigwedge_{i \in I} \theta_i(\bar{x}) \wedge \bigwedge_{j \in J} \neg \theta_j(\bar{x}) \right],$$

where  $I, J$  are disjoint finite subsets of  $\omega$ , would be a recursively axiomatizable 1-rich theory of trees.

Suppose the conclusion of Corollary 4.3 is replaced with: *Then there is a nonstandard model  $\mathcal{M} \models \text{PA}$  that is not recursively saturated such that  $\mathcal{M} \upharpoonright [\varphi]$  is a recursive tree.* I do not know if this modification of the corollary is true. There is even a stronger possible conclusion that I do not know to be false: *Then there is a prime nonstandard model  $\mathcal{M} \models \text{PA}$  for which  $\mathcal{M} \upharpoonright [\varphi]$  is a recursive tree.* The same problems exist when the hypothesis is strengthened to requiring that  $\mathcal{M} \upharpoonright [\varphi]$  is a tree for every  $\mathcal{M}$ . It might be interesting to resolve these modifications of Corollary 4.3.

Recall from the introduction that  $\text{DP} = \text{Th}[\exists z(y = x \cdot z)]$  is the theory of divisibility posets, and also that  $\text{DL}$  is the theory of divisibility lattices. Theorem 4.1 is inadequate for proving that the theory  $\text{DP}$  is 1-thin. For, if  $\mathcal{M} \models \text{PA}$  and  $\mathbf{K}$  is the class of all finite posets that are embeddable in the divisibility poset of  $\mathcal{M}$ , then  $\mathbf{K}$  is the class of all finite posets so that  $T_{\mathbf{K}} = \text{PO}$ , the theory of posets. It is rather easy to see that  $\text{PO}$  is not 1-thin. Incidentally, there is a single sentence in the language of posets having some poset as a model but none that is recursive<sup>5</sup>. However, Theorem 4.5, which is stronger than Theorem 4.1, is not only adequate for proving that  $\text{DP}$  is 1-thin, but is also for proving  $\text{DL}$  is 1-thin.

<sup>5</sup>It seems that the first example of a consistent sentence having no recursive model was given by Mostowski [28]. I thank Valentina Harizanov for bringing this reference to my attention.

If  $\mathbf{K}$  is a class of finite  $\mathcal{L}$ -structures, then a function  $f : \omega \rightarrow \omega$  is a *bound for algebraic closure* (abbreviation: *bac*) for  $\mathbf{K}$  if, whenever  $\mathfrak{A} \in \mathbf{K}$  and  $X \subseteq A$  is such that  $|X| \leq n$ , then there is  $\mathfrak{B} \in \mathbf{K}$  such that  $X \subseteq B$ ,  $\mathfrak{B} \subseteq \mathfrak{A}$  and  $|B| \leq f(n)$ . For example, the function  $n \mapsto n$  is a bac for  $\mathbf{K}$  iff  $\mathbf{K}$  is hereditary and  $\mathcal{L}$  has no constant symbols. For another example,  $n \mapsto 2^{2^n-1}$  is a bac for the class of finite distributive lattices.

**PROPOSITION 4.4.** *If  $\mathbf{K}$  is a strongly wqo class of finite structures, then  $\mathbf{K}$  has a bac.*

**PROOF.** Suppose that  $\mathbf{K}$  has no bac and that  $n < \omega$  is the least for which there is a problem. Thus, for each  $i < \omega$  there are  $\mathfrak{A}_i \in \mathbf{K}$  and  $X_i \subseteq A_i$  such that  $|X_i| = n$ ,  $|A_i| \geq i$  and there is no  $\mathfrak{B} \in \mathbf{K}$  such that  $\mathfrak{B} \subseteq \mathfrak{A}_i$  and  $X_i \subseteq B \neq A_i$ . We can arrange that  $|A_{i+1}| > |A_i|$  for each  $i < \omega$ . Let  $\mathfrak{A}_i^* = (\mathfrak{A}_i, a_{i0}, a_{i1}, \dots, a_{i(n-1)}) \in \mathbf{K}^{(n)}$  be an expansion of  $\mathfrak{A}_i$  such that  $X_i = \{a_{i0}, a_{i1}, \dots, a_{i(n-1)}\}$ . Since  $\mathbf{K}^{(n)}$  is wqo, there are  $i < j < \omega$  such that  $\mathfrak{A}_i^*$  is embeddable in  $\mathfrak{A}_j^*$ . Without loss of generality, assume  $\mathfrak{A}_i^* \subseteq \mathfrak{A}_j^*$ . But then,  $\mathfrak{A}_i \subseteq \mathfrak{A}_j$  and  $X_j = X_i \subseteq A_i \neq A_j$ , which is a contradiction.  $\dashv$

Suppose  $\mathbf{K}$  is a class of finite structures. If  $\mathbf{K}$  has a bac, then  $\forall_2 \cap \text{Th}(\mathbf{K}_\infty)$  is an axiomatization for  $T_{\mathbf{K}}$ , and  $\mathbf{K}_\infty$  is the class of models of  $T_{\mathbf{K}}$ . (There is a converse to this: if  $\mathbf{K}_\infty$  is the class of models of  $T_{\mathbf{K}}$ , then  $\mathbf{K}$  has a bac.) If  $f$  is a bac for  $\mathbf{K}$ , then there is a natural set of  $\forall_2$  sentences that axiomatize  $T_{\mathbf{K}}$ : the  $n$ -th sentence in this axiomatization is the one asserting that every set of at most  $n$  elements is in some substructure in  $\mathbf{K}$  having cardinality at most  $f(n)$ . Clearly, if both  $\mathbf{K}$  and  $f$  are recursive, then this axiomatization is a recursive axiomatization.

The next theorem improves Theorem 4.1; in fact, with the addition of the hypothesis that  $\mathbf{K}$  is hereditary, it becomes Theorem 4.1.

**THEOREM 4.5.** *Suppose that  $\mathbf{K}$  is a recursive, strongly wqo class of finite structures with a recursive bac. Then  $T_{\mathbf{K}}$  is 1-thin.*

**PROOF.** We indicate what modifications to the proof of Theorem 4.1 are needed.

Let  $f : \omega \rightarrow \omega$  be a recursive bac for  $\mathbf{K}$ , and then let  $b_0 < b_1 < b_2 < \dots$  be the sequence defined by  $b_0 = 0$  and  $b_{i+1} = f(b_i + 1)$ .

As in the proof of Theorem 4.1, we construct sequences  $\langle N_s : s < \omega \rangle$ ,  $\langle D_s : s < \omega \rangle$ , and  $\langle h_s : s < \omega \rangle$ . The only difference is that here we require that  $D_s$  be a  $(\mathbf{K}, \mathcal{L}_r)$ -diagram for some  $r$  such that  $b_s < r \leq b_{s+1}$ .

With these modifications, the proof of Theorem 4.1 can easily be successfully adapted to prove this theorem.  $\dashv$

The remark following Theorem 4.1 also applies to Theorem 4.5.

**REMARK.** In the previous proof, the model  $\mathfrak{A}'$  we constructed has the additional feature that  $\mathcal{D}(\mathfrak{A}')$  is  $\Delta_2^0$ .

When considering lattices with meet  $\wedge$  and join  $\vee$ , we will think of both  $\wedge$  and  $\vee$  as ternary relations. Every lattice  $(L, \wedge, \vee)$  has a partial ordering  $\leq$  defined by  $x \leq y \iff x \wedge y = x \iff x \vee y = y$ . If this ordering is a linear ordering, then  $(L, \wedge, \vee)$  is a *chain*. Every chain is a distributive lattice. We define a *box* to be a finite lattice that is the product of chains. Let  $\mathbf{B}$  be the class of all boxes. It follows from Higman's Theorem that  $\mathbf{B}$  is strongly wqo. (However,  $\mathbf{B}$  is not very wqo.) It is easy to see that  $\mathbf{B}$  is recursive and has a recursive bac; in fact, the function  $n \mapsto n^n$  is a bac for  $\mathbf{B}$ . We call a lattice in  $\mathbf{B}_\infty$  a *boxed* lattice, and we let BL be the theory of boxed lattices; thus,  $\text{BL} = T_{\mathbf{B}}$ . By identifying chains and linearly ordered sets, we can think of every linearly ordered set as being a model of BL. Clearly, BL is recursively axiomatizable (since  $\mathbf{B}$  is recursive and has a recursive bac). Thus, by Theorem 4.5, BL is 1-thin. Since DL is essentially a consequence of the complete theory Sk, it follows that DL is a decidable completion of BL and, therefore, that DL is 1-thin. Thus, by Corollary 2.6, we get the theorem from [37] that there are models of PA having recursive divisibility lattices, and we can even require that they be recursively saturated.

**COROLLARY 4.6.** *There is a recursively saturated model of PA having a recursive divisibility poset.*

We will end this section with a proof of Corollary 4.10, a result slightly weaker than Theorem 4.5. This proof will be useful in the next section.

The Hilbert-Bernays refinement of the Completeness Theorem says, essentially, that every recursively axiomatizable theory has a model  $\mathfrak{A}$  such that  $\mathcal{D}(\mathfrak{A})$  is  $\Delta_2^0$  [17]. Most any proof of the Completeness Theorem, if carefully carried out, yields this result. It is easy to improve this by requiring, in addition, that  $\mathfrak{A}$  be recursively saturated. An application of the Low Basis Theorem [11] yields a further improvement. Recall that a set  $X \subseteq \omega$  is low if its jump  $X'$  is  $\Delta_2^0$ .

**LEMMA 4.7.** *If  $T$  is recursively axiomatizable, then  $T$  has a recursively saturated model  $\mathfrak{A}$  such that  $\mathcal{D}(\mathfrak{A})$  is low.*

Let  $\mathbf{K}$  be a hereditary class of finite  $\mathcal{L}$ -structures. An  $\mathcal{L}$ -structure  $\mathfrak{B}$  is a *bound* for  $\mathbf{K}$  if  $\mathfrak{B} \notin \mathbf{K}$ ,  $\mathfrak{B}$  is finite, and every proper substructure of  $\mathfrak{B}$  is in  $\mathbf{K}$ . Notice that  $T_{\mathbf{K}}$  has a finite axiomatization iff  $\mathbf{K}$  has only finitely many bounds. The theorem of Pouzet [32], mentioned near the beginning of this section, can be restated as: if  $\mathbf{K}$  is a very wqo class of finite structures, then  $\mathbf{K}$  has only finitely many bounds. There is a stronger conclusion that can be made here, namely: if  $n < \omega$ , then  $\mathbf{K}^{(n)}$  has only finitely many bounds. We next discuss a closely related idea.

Let  $\mathfrak{A}$  be an  $\mathcal{L}$ -structure, and let  $a_0, a_1, \dots, a_{k-1}$  be a  $k$ -tuple of elements of  $A$ . Then, a  $\forall$ -basis for  $\bar{a}$  is a universal  $k$ -ary  $\mathcal{L}$ -formula  $\varphi(\bar{x})$  such that for any universal  $k$ -ary formula  $\sigma(\bar{x})$ ,  $\mathfrak{A} \models \sigma(\bar{a})$  iff  $\vdash \varphi(\bar{a}) \rightarrow \sigma(\bar{a})$ . A function

$\bar{a} \mapsto \varphi_{\bar{a}}(\bar{x})$  is a  $\forall$ -basis selector for  $\mathfrak{A}$  if  $\varphi_{\bar{a}}(\bar{x})$  is a  $\forall$ -basis whenever  $\bar{a}$  is a tuple from  $A$ .

Many structures have  $\forall$ -basis selectors. For example, if  $\mathbf{K}$  is a very wqo, hereditary class of finite structures and  $\mathfrak{A} \in \mathbf{K}_{\infty}$ , then  $\mathfrak{A}$  has a  $\forall$ -basis selector. To see this, consider  $a_0, a_1, \dots, a_{n-1} \in A$ , and then consider the class  $\mathbf{L}$  of all finite structures embeddable in  $(\mathfrak{A}, a_0, a_1, \dots, a_{n-1})$ . Clearly,  $\mathbf{L}$  is a very wqo, hereditary class of finite structures, so there is a universal sentence  $\varphi(\bar{a})$  that axiomatizes  $T_{\mathbf{L}}$ . But then,  $\varphi(\bar{x})$  is a  $\forall$ -basis for  $\bar{a}$ .

There is an immediate improvement to the discussion of the previous paragraph: if  $\mathbf{K}$  is a very wqo (not necessarily hereditary) class of finite structures and  $\mathfrak{A} \in \mathbf{K}_{\infty}$ , then  $\mathfrak{A}$  has a  $\forall$ -basis selector. For, if we let  $\mathbf{K}'$  be the smallest hereditary class containing  $\mathbf{K}$ , then  $\mathbf{K}'$  is a very wqo, hereditary class of finite structures, and  $\mathfrak{A} \in \mathbf{K}'_{\infty}$ .

The proof of the next lemma is completely straightforward.

LEMMA 4.8. *If  $\mathfrak{A}$  has a  $\forall$ -basis selector and  $\exists \cap \mathcal{D}(\mathfrak{A})$  is low, then  $\mathfrak{A}$  has a  $\Delta_2^0$   $\forall$ -basis selector.*  $\dashv$

LEMMA 4.9. *Suppose that  $\mathbf{K}$  is a recursive, strongly wqo class of finite structures with a recursive bac, and let  $\mathfrak{A} \models T_{\mathbf{K}}$ . If  $\mathfrak{A}$  is recursively saturated and has a  $\Delta_2^0$   $\forall$ -basis selector, then there is a recursive  $\mathfrak{B} \cong \mathfrak{A}$ .*

PROOF. Actually, we can get by with a property weaker than recursive saturation, which could be called *recursive existential saturation*, by just requiring that  $\mathfrak{A}$  realize all recursive consistent sets of existential formulas. That is, if  $\bar{a}$  is a  $k$ -tuple from  $A$  and  $\Phi(\bar{x}, y)$  is a recursive set of  $(k+1)$ -ary existential formulas such that  $\mathcal{D}(\mathfrak{A}) \cup \Phi(\bar{a}, y)$  is consistent, then there is  $b \in A$  such that  $\mathfrak{A} \models \Phi(\bar{a}, b)$ .

Now let  $\mathfrak{A} \models T_{\mathbf{K}}$  be recursively existentially saturated. We assume that  $\mathfrak{A}$  is infinite, as otherwise the conclusion is trivial. Let  $\bar{a} \mapsto \varphi_{\bar{a}}(\bar{x})$  be a  $\Delta_2^0$   $\forall$ -basis selector. Notice that  $\varphi_{\emptyset}$  is an axiomatization for  $\forall \cap \text{Th}(\mathfrak{A})$ , so that  $\exists \cap \text{Th}(\mathfrak{A})$  is recursive. Thus, without loss of generality, we can assume that  $\mathbf{K}$  is the class of finite substructures embeddable in  $\mathfrak{A}$ , and we still have that  $\mathbf{K}$  is a recursive, strongly wqo class of finite structures with a recursive bac.

Since  $\mathbf{K}$  is recursive and strongly wqo, we have that each  $\mathbf{K}^{(n)}$  is recursive and wqo. We next make two observations about maximal filters in recursive wqo sets in general.

Suppose  $(Q, \preceq)$  is a quasi order. A subset  $F \subseteq Q$  is a *filter* if (1) whenever  $x \preceq y \in F$ , then  $x \in F$ , and (2) whenever  $x, y \in F$ , there is  $z \in F$  such that  $x, y \preceq z$ . A filter  $F \subseteq Q$  is *maximal* if there is no filter  $G \subseteq Q$  of which  $F$  is a proper subset. The first observation is:

*A wqo  $(Q, \preceq)$  has only finitely many maximal filters.*

For, suppose that  $F_0, F_1, F_2, \dots$  are infinitely many distinct maximal filters. Then, for each  $n < \omega$ ,  $F_n \setminus \bigcup_{i < n} F_i \neq \emptyset$ . To get  $b_n \in F_n \setminus \bigcup_{i < n} F_i$ , let  $a_i \in F \setminus F_i$ , for  $i < n$  and then let  $a \in F_n$  be such that  $a_0, a_1, \dots, a_{n-1} \preceq b_n$ .

By applying Ramsey's Theorem and taking a subsequence if needed, we can assume that  $b_0 \trianglelefteq b_1 \trianglelefteq b_2 \trianglelefteq \dots$ . For each  $n < \omega$ , choose  $c_n$  such that  $b_n \trianglelefteq c_n \in F_n$  and  $c_n, b_{n+1}$  are *incompatible*, that is there is no  $x \in Q$  such that  $c_n, b_{n+1} \trianglelefteq x$ . Clearly,  $c_0, c_1, c_2, \dots$  is an infinite antichain, which is impossible in a wqo.

The second observation is:

*If  $(Q, \trianglelefteq)$  is a recursive wqo, then every maximal filter is recursively enumerable.*

To see this, let  $F_0, F_1, \dots, F_n$  be the finitely many distinct maximal filters. Then choose  $a_i \in F_i$ , for  $i \leq n$ , so that  $a_0, a_1, \dots, a_n$  are pairwise incompatible. Then,  $F_i$  is the set of  $x \in Q$  that are compatible with  $a_i$ .

Now consider a  $k$ -tuple  $\bar{a}$  from  $A$ . Let  $F_{\bar{a}} \subseteq \mathbf{K}^{(k)}$  be the class of those structures in  $\mathbf{K}^{(k)}$  that are embeddable in  $(\mathfrak{A}, \bar{a})$ . Then  $F$  is a filter. Since  $\mathfrak{A}$  is recursively existentially saturated, the second of the prior observations implies that if  $F \subseteq \mathbf{K}^{(k)}$  is a maximal filter, then there is a  $k$ -tuple  $\bar{a}$  from  $A$  such that  $F = F_{\bar{a}}$ . It then easily follows that:

(♠) *Suppose  $\bar{a}$  is a  $k$ -tuple from  $A$ . Then there are finitely many  $b_0, b_1, \dots, b_n \in A$  such that for any  $c \in A$ , there is  $i \leq n$  such that  $\vdash \varphi_{\bar{a}c}(\bar{x}) \rightarrow \varphi_{\bar{a}b_i}(\bar{x})$ .*

Recall that we are assuming that  $\mathfrak{A}$  is infinite, so we assume, without loss of generality, that  $A = \omega$ . If  $N \subseteq \omega$ , then let  $[N]^{<\omega}$  be the set of all increasing, finite sequences from  $N$ . Then there is a recursive, doubly-indexed set  $\langle \varphi_{\bar{a}}^s(\bar{x}) : s < \omega, \bar{a} \in [\omega]^{<\omega} \rangle$  of universal formulas such that  $\lim_s \varphi_{\bar{a}}^s(\bar{x}) = \varphi_{\bar{a}}(\bar{x})$ . For convenience, we assume that  $\varphi_{\langle 0 \rangle}^s(\bar{x}) = \varphi_{\langle 0 \rangle}(\bar{x})$  for all  $s < \omega$ . If  $s < \omega$ ,  $h : N \rightarrow \omega$  and  $\bar{a} = \langle a_0, a_1, \dots, a_{k-1} \rangle \in [N]^{<\omega}$ , then let

$$\sigma_{\bar{a},h}^s = \varphi_{\bar{a}}^s(c_{h(a_0)}, c_{h(a_1)}, \dots, c_{h(a_{k-1})}),$$

and let

$$T(h, s) = \{ \sigma_{\bar{a},h}^s : \bar{a} \in [N]^{<\omega} \}.$$

Let  $b_0 < b_1 < b_2 < \dots$  be a sequence as in the proof of Theorem 4.5. That is, let  $p : \omega \rightarrow \omega$  be a recursive bac for  $\mathbf{K}$  and then let  $b_0 = 0$  and  $b_{i+1} = p(b_i + 1)$ . Also, recall what is meant by a  $(\mathbf{K}, \mathcal{L}_r)$ -diagram  $D$ .

We will effectively construct sequences  $\langle D_s : s < \omega \rangle$ ,  $\langle \ell_s : s < \omega \rangle$ ,  $\langle f_s : s < \omega \rangle$  and  $\langle g_s : s < \omega \rangle$  where  $D_s$  is a  $(\mathbf{K}, \mathcal{L}_r)$ -diagram for some  $r$  such that  $b_s < r \leq b_{s+1}$ ,  $\ell_s \leq s$ ,  $f_s : \{n : 2n < \ell_s\} \rightarrow s$  is a one-to-one function,  $g_s : \{n : 2n + 1 < \ell_s\} \rightarrow s$  is a one-to-one function, and  $f_s$  and  $g_s$  are partial inverses of each other. (By this we mean that  $f_s(g_s(n)) = n$  whenever  $g_s(n), f_s(g_s(n))$  are defined, and that  $g_s(f_s(n)) = n$  whenever  $f_s(n), g_s(f_s(n))$  are defined. Alternatively, we could say that  $f_s$  and  $g_s$  are partial inverses of each other if  $f_s \cup g_s^{-1}$  is a one-to-one function.)

Furthermore, the following will hold for each  $s < \omega$ :

- (0)  $D_s \cup \{ \varphi_{\langle 0 \rangle}(c_0) \}$  is consistent;
- (1)  $D_s \subseteq D_{s+1}$ ;

- (2)  $h_s = f_s \cup g_s^{-1}$  is a one-to-one function, and  $D_s \cup T(h_s, s)$  is consistent.

We begin by letting  $\ell_0 = 0$ ,  $f_0 = g_0 = \emptyset$ , and  $D_0$  be some  $(\mathbf{K}, \mathcal{L}_r)$ -diagram where  $0 = b_0 < r \leq b_1$ .

At the onset of stage  $s$ , we have  $D_s$ ,  $\ell_s$ ,  $f_s$  and  $g_s$  satisfying (0)–(2). Let  $\ell \leq \ell_s$  be the largest such that there are  $r$  such that  $b_{s+1} < r \leq b_{s+2}$ , a  $(\mathbf{K}, \mathcal{L}_r)$ -diagram  $D \supseteq D_s$ , functions  $f : \{n : 2n \leq \ell\} \rightarrow s+1$  and  $g : \{n : 2n+1 \leq \ell\} \rightarrow s+1$  such that:

- (2')  $h = f \cup g^{-1}$  is a one-to-one function and  $D \cup T(h, s+1)$  is consistent.  
 (3') if  $2n < \ell$ , then  $f(n) = f_s(n)$ ; and if  $2n+1 < \ell$ , then  $g(n) = g_s(n)$ ;  
 (4') if  $N = \{n : 2n \leq \ell\} \cup \{g(n) : 2n+1 \leq \ell\}$  and  $\bar{a} \in [N]^{<\omega}$ , then  $\varphi_{\bar{a}}^{s+1}(\bar{x}) = \varphi_{\bar{a}}^s(\bar{x})$ ;

First, observe that there is such an  $\ell$  because we can let  $\ell = 0$ ,  $f = \{\langle 0, 0 \rangle\}$ , and  $g = \emptyset$ , and then let  $D \supseteq D_s$  be any  $(\mathbf{K}, \mathcal{L}_r)$ -diagram consistent with  $\varphi_{\langle 0 \rangle}(c_0)$  such that  $b_{s+1} < r \leq b_{s+2}$ . Trivially, (2') – (4') hold. Next, observe that  $\ell$  is effectively defined, and that  $f, g, D$  can also be effectively defined. We will be more explicit about the choice of  $f$  and  $g$ : if  $\ell = 2n$ , then we let  $f(n)$  be as small as possible; and if  $\ell = 2n+1$ , then we let  $g(n)$  be as small as possible. Having defined  $\ell, f, g, D$ , let  $D_{s+1} = D$ ,  $\ell_{s+1} = \ell + 1$ ,  $f_{s+1} = f$ , and  $g_{s+1} = g$ .

Thus, we have effectively constructed the sequences  $\langle D_s : s < \omega \rangle$ ,  $\langle \ell_s : s < \omega \rangle$ ,  $\langle f_s : s < \omega \rangle$  and  $\langle g_s : s < \omega \rangle$  such that, for each  $s < \omega$ , (0) – (5) hold, where:

- (3) if  $2n+1 < \ell_s$ , then  $f_{s+1}(n) = f_s(n)$ ; and if  $2n+2 < \ell_s$ , then  $g_{s+1}(n) = g_s(n)$ ;  
 (4) if  $N = \{n : 2n < \ell_{s+1}\} \cup \{g_{s+1}(n) : 2n+1 < \ell_{s+1}\}$  and  $\bar{a} \in [N]^{<\omega}$ , then  $\varphi_{\bar{a}}^{s+1}(\bar{x}) = \varphi_{\bar{a}}^s(\bar{x})$ ;  
 (5)  $1 \leq \ell_{s+1} \leq \ell_s + 1$ .

For each  $s < \omega$ , let  $h_s = f_s \cup g_s^{-1}$ ,  $N_s = \text{dom } h_s = \{n : 2n < \ell_s\} \cup \{g_s(n) : 2n+1 < \ell_s\}$  and  $M_s = \text{ran } h_s = \{h_s(n) : n \in N_s\}$ .

Let  $D = \bigcup_s D_s$ , and let  $\mathfrak{B}$  be the structure whose diagram is  $D$ . Clearly,  $\mathfrak{B}$  is recursive. It remains to show that  $\mathfrak{B} \cong \mathfrak{A}$ , which we will show by obtaining a permutation  $f : \omega \rightarrow \omega$  such that the function  $i \mapsto c_{f(i)}$  is an isomorphism from  $\mathfrak{A}$  to  $\mathfrak{B}$ . The key fact that is needed here is the following claim.

CLAIM.  $\lim_s \ell_s = \infty$ .

For, having this claim, we let  $f = \lim_s f_s$  and  $g = \lim_s g_s$  which, by (3), are well defined functions  $f : \omega \rightarrow \omega$  and  $g : \omega \rightarrow \omega$ , and by (2) are inverses of each other. Hence,  $f : \omega \rightarrow \omega$  is a permutation. It then follows from (2) that  $i \mapsto c_{f(i)}$  is an isomorphism from  $\mathfrak{A}$  to  $\mathfrak{B}$ .

To prove the claim, suppose it is false. Let  $m < \omega$  be the least such that  $\{s < \omega : \ell_s = m\}$  is infinite. Then  $m > 0$  by (5). There are two cases to consider, depending on the parity of  $m$ .

*The odd case:*  $m = 2n + 1$ . Let  $t < \omega$  be such that  $\ell_t = m$  and  $\ell_s \geq m$  whenever  $t \leq s < \omega$ . Moreover, using  $(\spadesuit)$ , we can assume that  $t$  is large enough so that

- $(\clubsuit)$  whenever  $\bar{a} \in [M_s]^{<\omega}$  and  $c \in A$ , then there is  $b < t$  such that  $\vdash \varphi_{\bar{a}c}(\bar{x}) \rightarrow \varphi_{\bar{a}b}(\bar{x})$ .

Let  $s > t$  be such that  $\ell_s = m$  and

- $(\diamond)$  whenever  $\bar{a} \in [\{0, 1, 2, \dots, t-1\}]^{<\omega}$  and  $r \geq s$ , then  $\varphi_{\bar{a}}^r(\bar{x}) = \varphi_{\bar{a}}(\bar{x})$ .

Let  $b = f_s(n)$ . Since, in the construction,  $f_s(n)$  was chosen to be minimal,

$(\clubsuit)$  implies that  $b < t$ . Then  $(\diamond)$  implies that  $T(h_s, s) = T(h_s, r)$  for all  $r > s$ , so that  $f_r(n) = b$  for all  $r > s$ . But then  $\ell_r > m$  for all  $r > s$ , a contradiction.

*The even case:*  $m = 2n + 2$ . Let  $t < \omega$  be such that  $\ell_t = m$  and  $\ell_s \geq m$  whenever  $t \leq s < \omega$ . Let  $s > t$  be such that  $\ell_{s+1} = m$  and

- $(\heartsuit)$  whenever  $\bar{a} \in [M_s \cup \{n\}]^{<\omega}$  and  $r \geq s$ , then  $\varphi_{\bar{a}}^r(\bar{x}) = \varphi_{\bar{a}}(\bar{x})$ .

Then  $(\heartsuit)$  implies that  $T(h_s, s) = T(h_s, r)$  for all  $r > s$ , so that  $g_r(n) = g_s(n)$  for all  $r > s$ . But then  $\ell_r > m$  for all  $r > s$ , a contradiction.  $\dashv$

As a corollary, we get a weaker version of Theorem 4.5. But the proof yields an additional feature that will be useful in the next section.

**COROLLARY 4.10.** *Suppose that  $\mathbf{K}$  is a recursive, very wqo class with a recursive bac. Then  $T_{\mathbf{K}}$  is 1-thin.*

**PROOF.** Let  $\mathcal{L}$  be the language for  $\mathbf{K}$ , and let  $T \supseteq T_{\mathbf{K}}$  be a recursively axiomatizable  $\mathcal{L}$ -theory. By Lemma 4.7, let  $\mathfrak{A} \models T$  be recursively saturated such that  $\mathcal{D}(\mathfrak{A})$  is low. The conditions on  $\mathbf{K}$  imply that  $\mathfrak{A}$  has a  $\forall$ -basis selector, so Lemma 4.8 implies that  $\mathfrak{A}$  has one that is  $\Delta_2^0$ . Then, by Lemma 4.9, there is a recursive  $\mathfrak{B} \cong \mathfrak{A}$ , which, by Corollary 2.4(2), shows that  $T_{\mathbf{K}}$  is 1-thin.  $\dashv$

**REMARK.** A feature of this proof of Corollary 4.10 is that, not only is the constructed model  $\mathfrak{A}$  recursive and recursively saturated, but also its theory  $\text{Th}(\mathfrak{A})$  is low. By the remarks following the proofs of Theorems 4.1 and 4.5, we can, instead, get  $\mathcal{D}(\mathfrak{A})$  to be  $\Delta_2^0$ . This suggests the following question.

**QUESTION 4.11.** In proving Theorem 4.5, is it possible to get a recursive, recursively saturated model  $\mathfrak{A}$  such that  $\mathcal{D}(\mathfrak{A})$  is low?

**§5. More about LO.** Peretyatkin [31] proved that every recursively axiomatizable theory of linearly ordered sets has a recursive model. This result has been improved in two different ways: to trees (Ershov [6]) and to  $\Sigma_2^0$  theories (Lerman and Schmerl [22]). As already noted, a consequence of Theorem 4.1 is that the theory of trees is 1-thin, slightly improving Ershov's theorem. Another improvement, coming from the discussion prior to Corollary 4.6, is that BL is 1-thin. In this section, we make an improvement in another direction by proving that the theory LO is 2-thin, slightly improving the result in [22]. There is undoubtedly a proof of this latter fact along the lines of the proof

in [22]. However, the proof we give for LO is a different proof that is simpler than the one in [22] as it does not require an infinite-injury priority argument.

In order to get the 2-thinness of LO, we will first improve upon the 1-thinness of LO in another way by obtaining models having recursive  $\forall$ -basis selectors. In general, if  $\mathfrak{A}$  has a recursive  $\forall$ -basis selector, then not only is  $\mathfrak{A}$  recursive, but also  $\forall \cap \mathcal{D}(\mathfrak{A})$  is recursive. In general, the converse is not true. For example, one can rather easily construct a linear order  $\mathfrak{B} = (B, <)$  having order type  $\omega + (\omega^* + \omega) \cdot \eta$  (which is the order type of every countable, nonstandard model of PA) in which the order and successor relations are recursive but the binary relation that holds between  $x, y \in B$  iff there are only finitely many points between them is not recursive. Such a  $\mathfrak{B}$  does not have a recursive  $\forall$ -basis selector despite its being a decidable, saturated linearly ordered set.

**THEOREM 5.1.** *Suppose that  $T \supseteq \text{LO}$  is a recursively axiomatizable  $\{<\}$ -theory. Then  $T$  has a recursively saturated model with a recursive  $\forall$ -basis selector.*

**PROOF.** Let  $\mathcal{L} = \{<\}$ . An alternate and equivalent formulation of this theorem (cf. Corollary 2.4(2) with  $n = 1$ ) is:

*Suppose  $\mathcal{L}' \supseteq \mathcal{L}$  is finite and  $T' \supseteq \text{LO}$  is a recursively axiomatizable  $\mathcal{L}'$ -theory. Then  $T'$  has a model  $\mathfrak{A} = (A, <, \dots)$  such that  $(A, <)$  has a recursive  $\forall$ -basis selector.*

We will prove this alternate formulation. Its proof is modeled after the proof of Theorem 4.1. We assume that  $T'$  has no finite models, as otherwise we can let  $\mathfrak{A}$  be a finite model. Without loss of generality, we assume that every model of  $T'$  has a first and a last element.

Let  $E_0, E_1$  be new unary relation symbols, and let  $S_\infty, S_0, S_1, S_2, \dots$  be new binary relation symbols. For  $r < \omega$ , let  $\mathcal{L}(r) = \{<, E_0, E_1, S_\infty, S_0, S_1, \dots, S_{r-1}\}$  and let  $\mathcal{L}(\omega) = \bigcup_{r < \omega} \mathcal{L}(r)$ . (We are making an exception here of our convention that languages are finite.) Let  $\mathbf{K}(r)$  be the class of finite  $\mathcal{L}(r)$ -structures  $(A, <, E_0, E_1, S_\infty, S_0, S_1, \dots, S_{r-1})$  such that  $(A, <)$  is a finite linearly ordered set with at least two elements and for  $a, b, c \in A$ , the following hold:

- (1)  $a \in E_0$  iff  $a$  is the first element;
- (2)  $b \in E_1$  iff  $b$  is the last element;
- (3) if  $a \in E_0$  and  $b \in E_1$ , then  $aS_\infty b$ ;
- (4) if  $aS_i b$  for some  $i < r$  or  $aS_\infty b$ , then  $a < b$ ;
- (5) if  $a < b$ , then either  $aS_\infty b$  or there is a unique  $i < r$  such that  $aS_i b$ ;
- (6) if either  $a < bS_\infty c$  or  $aS_\infty b < c$ , then  $aS_\infty c$ ;
- (7) if  $i, j < r$  and  $aS_i bS_j c$ , then  $i + j + 1 < r$  and  $aS_{i+j+1} c$ ;
- (8) if  $aS_\infty b$ , then  $aS_i b$  for no  $i < r$ .

[The intuition here is that if  $i < k$  and  $aS_i b$  then there is a potential that  $|\{x : a < x < b\}| = i.$ ]

If  $n < \omega$ , then  $K^{(n)}(r)$  is the class of all expansions  $(\mathfrak{A}, a_0, a_1, \dots, a_{n-1})$ , where  $\mathfrak{A} \in K(r)$ . Let  $K^{(n)}(\omega) = \bigcup_{r < \omega} K^{(n)}(r)$ . Given  $\mathfrak{A}^* = (\mathfrak{A}, a_0, a_1, \dots, a_{n-1})$  and  $\mathfrak{B}^* = (\mathfrak{B}, b_0, b_1, \dots, b_{n-1})$  in  $K^{(n)}(\omega)$ , we say that  $e : \mathfrak{A}^* \rightarrow \mathfrak{B}^*$  is an *enlarging* if  $\mathfrak{A} \in K(r)$ ,  $\mathfrak{B} \in K(s)$ , where  $r \leq s$ ,  $e$  is an embedding of  $(\mathfrak{A} \upharpoonright \mathcal{L}_0, a_0, a_1, \dots, a_{n-1})$  into  $(\mathfrak{B} \upharpoonright \mathcal{L}_0, b_0, b_1, \dots, b_{n-1})$  and whenever  $x, y \in A$ ,  $i, j \leq \infty$ ,  $\mathfrak{A} \models x S_i y$  and  $\mathfrak{B} \models e(x) S_j e(y)$ , then  $i \leq j$ . If there is an enlarging  $e : \mathfrak{A}^* \rightarrow \mathfrak{B}^*$ , then  $\mathfrak{A}^*$  is *enlargeable* in  $\mathfrak{B}^*$ .

We consider  $K^{(n)}(\omega)$  to be quasi-ordered by *enlargeability* (that is, the existence of an enlarging). The following is a consequence of Higman's Theorem.

**PROPOSITION 5.1.1.** *If  $n < \omega$ , then  $K^{(n)}(\omega)$  is wqo by enlargeability.*

We first prove this proposition for  $n = 0$ . Let  $Q = \{0, 1, 2, \dots, \infty\}$  be well-ordered by  $0 \trianglelefteq 1 \trianglelefteq 2 \trianglelefteq \dots \trianglelefteq \infty$ , so by Higman's Theorem,  $Q^{<\omega}$  is wqo. If  $\mathfrak{A} = (A, <, E_0, E_1, S_\infty, S_0, S_1, S_2, \dots) \in K(\omega)$  has the  $n + 1$  elements  $a_0 < a_1 < \dots < a_n$ , then associate with it the  $n$ -tuple  $t(\mathfrak{A}) = \langle t_0, t_1, \dots, t_{n-1} \rangle \in Q^n$ , where  $\mathfrak{A} \models a_i S_{t_i} a_{i+1}$ . Let  $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$  be an infinite sequence of structures in  $K(\omega)$ . Since  $Q^{<\omega}$  is wqo, there are  $i < j < \omega$ , such that  $t(\mathfrak{A}_i) \trianglelefteq t(\mathfrak{A}_j)$ . Clearly, then,  $\mathfrak{A}_i$  is enlargeable in  $\mathfrak{A}_j$ , so that  $K(\omega)$  is wqo.

Now consider  $n > 0$ . If  $\mathfrak{A}^* = (\mathfrak{A}, a_1, a_2, \dots, a_n) \in K^{(n)}(\omega)$ , where we are assuming without loss of generality that  $a_0, a_{n+1}$  are its first and last element and that  $a_0 < a_1 < \dots < a_n < a_{n+1}$ , then associate with  $\mathfrak{A}$  the  $(n + 1)$ -tuple  $\langle \mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n \rangle$  of substructures of  $\mathfrak{A}$ , where the universe of  $\mathfrak{A}_i$  is the interval  $[a_i, a_{i+1}]$ . With this association, since  $K(\omega)$  is wqo, Higman's Theorem implies that  $K^{(n)}(\omega)$  is wqo. This proves the proposition.

As we have done before, we let  $c_0, c_1, c_2, \dots$  be a supply of new and distinct constant symbols. If  $r \leq \omega$  and  $s < \omega$ , then  $\mathcal{L}_s(r) = \mathcal{L}(r) \cup \{c_0, c_1, \dots, c_{s-1}\}$ . If  $r, s < \omega$ , then by an  $s$ -diagram we will mean what we referred to in §4 as a  $(K(s), \mathcal{L}_{s+2}(s))$ -diagram.

As in the proof of Theorem 4.1, let  $\mathcal{L}'_s = \mathcal{L}' \cup \{c_0, c_1, \dots, c_{s-1}\}$ , and let  $\mathcal{L}'_\omega = \bigcup_{s < \omega} \mathcal{L}'_s$ . Let  $\varphi_0(x), \varphi_1(x), \varphi_2(x), \dots$  be a recursive list of all  $\mathcal{L}'_\omega$ -formulas having  $x$  as the only free variable such that each  $\varphi_i(x)$  is an  $\mathcal{L}'_i$ -formula, and let  $\varphi_0(x) = E_0(x)$  and  $\varphi_1(x) = E_1(x)$ . If  $I \subseteq \omega$  and  $h : I \rightarrow \omega$ , then  $H(h)$  is just as in the proof of Theorem 4.1. Let  $\sigma_0, \sigma_1, \sigma_2, \dots$  be a recursive list of all  $\mathcal{L}'_\omega$ -sentences that can be logically deduced from  $T'$ , where each  $\sigma_i$  is an  $\mathcal{L}'_i$ -sentence. If  $\Gamma$  is a set of  $\mathcal{L}'_\omega$ -sentences, then we say that  $\Gamma$  is *s-consistent* if no contradiction can be derived from  $\Gamma \cup \{\sigma_0, \sigma_1, \dots, \sigma_{s-1}\}$  in  $s$  steps.

Suppose  $D \subseteq D'$ , where  $D$  is an  $s$ -diagram and  $D'$  is an  $(s + 1)$ -diagram. Then there are uniquely determined  $i, j \leq s + 1$  such if  $k \leq s + 2$ , then  $\{c_i < c_k, c_k < c_j\} \subseteq D'$  iff  $k = s + 2$ . Let us say that  $D'$  is a  $(p, q)$ -cover of  $D$  if  $\{c_i S_p c_{s+2}, c_{s+2} S_q c_j\} \subseteq D'$ , where  $p, q \in \{0, 1, 2, \dots, s\} \cup \{\infty\}$ .

Let  $\delta_k(x, y)$  be the formula asserting that  $x < y$  and that there are at least  $k$  points between  $x$  and  $y$ ; thus,

$$\delta_k(x, y) = \exists x_0, x_1, \dots, x_{k-1} [x < x_0 < x_1 < \dots < x_{k-1} < y].$$

Given an  $s$ -diagram  $D$ , for each  $i, j \leq s+1$ , let  $U_{ij}(D)$  be the set of sentences such that:

- if  $c_i < c_j$  is not in  $D$ , then  $U_{ij}(D) = \emptyset$ ;
- if  $k < s$  and  $c_i S_k c_j$  is in  $D$ , then  $U_{ij}(D) = \{\delta_k(c_i, c_j), \neg\delta_{k+1}(c_i, c_j)\}$ ;
- if  $c_i S_\infty c_j$  is in  $D$ , then  $U_{ij}(D) = \{\delta_k(c_i, c_j) : k < \omega\}$ .

Then let  $U(D) = D \cup \{\forall x[E_0(x) \leftrightarrow x = c_0], \forall x[E_1(x) \leftrightarrow x = c_1], \forall x[c_0 \leq x \leq c_1]\} \cup \{U_{ij}(D) : i, j \leq s+1\}$ . For  $s < \omega$ , let  $U^s(D)$  be the set of those sentences in  $U(D)$  having no more than  $s+1$  quantifiers.

We will construct recursive sequences  $\langle N_s : s < \omega \rangle$ ,  $\langle h_s : s < \omega \rangle$  and  $\langle D_s : s < \omega \rangle$  by stages. At stage  $s$ , we will define  $N_s \leq s+2$ , a function  $h_s : N_s \rightarrow s+3$  and an  $s$ -diagram  $D_s$  so that:

- (0)  $T' \cup U(D_s)$  is consistent;
- (1)  $H(h_s) \cup U^s(D_s)$  is  $s$ -consistent.

We begin by letting  $N_0 = 2$ ,  $h_0 : 2 \rightarrow 3$  be the identity function and  $D_0$  be the 0-diagram such that  $D_0 \supseteq \{E_0(c_0), E_1(c_1), c_0 < c_1, c_0 S_\infty c_1\}$ .

At the onset of stage  $s+1$ , we have  $N_s \leq s+3$ ,  $h_s : N_s \rightarrow s+2$  and an  $s$ -diagram  $D_s$  such that (0) and (1) hold. Let  $t \leq N_s$  be the largest for which there are an  $(s+1)$ -diagram  $D_{s+1} \supseteq D_s$  and a function  $h_{s+1} : t+1 \rightarrow s+4$  such that  $h_{s+1} \upharpoonright t = h_s \upharpoonright t$ ,  $T' \cup U(D_{s+1})$  is consistent, and  $H(h_{s+1}) \cup U^{s+1}(D_{s+1})$  is  $(s+1)$ -consistent.

Observe that  $t$  is well defined and  $t \geq 2$ . For, since (0) holds, we can let  $D_{s+1} \supseteq D_s$  be any  $(s+1)$ -diagram such that  $U(D_{s+1})$  is consistent with  $T'$ . (This can be done by letting  $D_{s+1}$  be an  $(\infty, \infty)$ -cover of  $D_s$  such that  $c_i S_\infty c_j$  is in  $D_s$ , and  $c_i < c_k < c_j$  is in  $D_{s+1}$  iff  $k = s+2$ .) Then, letting  $t = 2$  and  $h_{s+1} : 3 \rightarrow s+4$  be the identity function, we clearly have  $H(h_{s+1}) \cup U^{s+1}(D_{s+1})$  is  $(s+1)$ -consistent. Notice that if  $t < N_s$ , then  $h_{s+1}(t) \neq h_s(t)$  as otherwise the maximality of  $t$  would be contradicted by using the function  $h_{s+1} \cup \{ \langle t+1, c_{s+4} \rangle \}$ .

Having  $t$ , let  $N_{s+1} = t+1$ , and then choose  $h_{s+1}$  and  $D_{s+1}$  so that:

- if it is possible that  $D_{s+1}$  is an  $(\infty, \infty)$ -cover of  $D_s$ , then  $D_{s+1}$  is an  $(\infty, \infty)$ -cover of  $D_s$ ;
- if it is not possible that  $D_{s+1}$  is an  $(\infty, \infty)$ -cover of  $D_s$ , but it is possible that  $D_{s+1}$  is an  $(\infty, q)$ -cover of  $D_s$ , then  $D_{s+1}$  is an  $(\infty, q)$ -cover of  $D_s$ , where  $q$  is as large as possible;
- if it is not possible that  $D_{s+1}$  is an  $(\infty, q)$ -cover of  $D_s$ , but it is possible that  $D_{s+1}$  is a  $(p, \infty)$ -cover of  $D_s$ , then  $D_{s+1}$  is a  $(p, \infty)$ -cover of  $D_s$ , where  $p$  is as large as possible.

For each  $s < \omega$ , the following hold:

- (2)  $D_{s+1} \supseteq D_s$ ;
- (3)  $3 \leq N_{s+1} \leq N_s + 1$ ,  $h_{s+1} \upharpoonright (N_{s+1} - 1) \subseteq h_s$ , and  $h_{s+1}(N_{s+1} - 1) \neq h_s(N_{s+1} - 1)$  if  $N_{s+1} \leq N_s$ .

The crucial fact needed to complete the proof is the following claim.

CLAIM.  $\lim_s N_s = \infty$ .

For, having this claim, we let  $h = \lim_s h_s$ , which by (3), is now well defined and  $h : \omega \rightarrow \omega$ . Let  $D = \bigcup D_s$  and  $T'' = D \cup H(h) \cup T'$ . It follows from (1) that  $T''$  is  $s$ -consistent for all  $s < \omega$  and, therefore,  $T''$  is consistent. Clearly,  $D$  is recursive by (2). Let  $\mathfrak{A}'$  be the Henkin model of  $T''$ . Then its reduct  $\mathfrak{A} = \mathfrak{A}' \upharpoonright \mathcal{L}$  is a recursive linearly ordered set that has a recursive  $\forall$ -basis selector. It is enough to get a  $\forall$ -basis selector just for pairs of elements in  $\mathfrak{A}$ , and whenever  $a, b \in A$  and  $a < b$ , then the unique  $i \leq \infty$  such that  $aS_i b \in D$  determines the  $\forall$ -baiss of  $\langle a, b \rangle$ .

It remains to prove the claim. Suppose the claim is false. By (3),  $N_s \geq 3$  if  $s > 0$ , so we can let  $m = \liminf_s N_s - 1$ . Let  $r < \omega$  be such that  $N_r = m + 1$  and  $N_s \geq m + 1$  for all  $s \geq r$ . By (3),  $h_s \upharpoonright m = h_r \upharpoonright m$  whenever  $r \leq s < \omega$ ; in particular,  $h_s(0) = h_0(0) = 0$  and  $h_s(1) = h_0(1) = 1$  for all  $s < \omega$ . Then let  $r < s_0 < s_1 < s_2 < \dots$  be such that  $N_{s_i} = m + 1$  for all  $i < \omega$ . Let  $\mathfrak{A}_i$  be the  $\mathcal{L}(s_i - 1)$ -structure whose diagram is  $D_{s_i-1}$ . Let  $a_k = c_{h_r(k)}$  for  $k < m$ , and  $b_i = c_{h_{s_i-1}(m)}$ . Then, let  $\mathfrak{A}_i^* = (\mathfrak{A}_i, a_0, a_1, \dots, a_{m-1}, b_i) \in \mathbf{K}^{(m+1)}(s_i - 1)$ . [This definition of  $\mathfrak{A}_i^*$  needs that  $h_{s_i-1}(m) \in A_i$ , so we assume this to be the case for infinitely many  $i$ . If not, that is if  $h_{s_i-1}(m) = s_i + 2$  for all but finitely many  $i$ , then there is a very similar argument that works.]

Since  $\mathbf{K}^{(m+1)}(\omega)$  is wqo, there are  $i < j$  such that  $\mathfrak{A}_i^*$  is enlargeable in  $\mathfrak{A}_j^*$ . Moreover, there are  $i_0 < i_1 < i_2 < \dots$  such that each  $\mathfrak{A}_{i_k}^*$  is enlargeable in  $\mathfrak{A}_{i_{k+1}}^*$ . We then might as well assume that  $\mathfrak{A}_i^*$  is enlargeable in  $\mathfrak{A}_{i+1}^*$ . By composing these enlargings, we get enlargings  $e_i : \mathfrak{A}_0^* \rightarrow \mathfrak{A}_i^*$  for each  $i < \omega$ . Hence,  $D_{s_0-1} \cup H(h_{s_0-1} \upharpoonright (m+1))$  is  $(s_i - 1)$ -consistent for each  $i < \omega$ , and, therefore,  $D_{s_0-1} \cup H(h_{s_0-1} \upharpoonright (m+1)) \cup T'$  is consistent. But then, there must be an  $\mathcal{L}_{s_0}$ -diagram  $D'$  and  $h' : m+2 \rightarrow s_0$  such that  $h' \upharpoonright (m+1) = h_{s_0-1} \upharpoonright (m+1)$  and  $D' \cup H(h') \cup T'$  is consistent. Therefore,  $N_{s_0} \geq m + 2$ , a contradiction that proves the claim and the theorem.  $\dashv$

The given proof of Theorem 5.1 is easily adapted to prove a relativized version. In particular: *If  $T \supseteq \text{LO}$  is a  $\Sigma_n^0 \{<\}$ -theory, then  $T$  has a  $\Delta_n^0$ -recursively saturated model with a  $\Delta_n^0 \forall$ -basis selector.*

COROLLARY 5.2. *The theory of linearly ordered sets is 2-thin.*

PROOF. This corollary follows from Lemma 4.9 and the appropriate (letting  $n = 2$ ) relativized version of Theorem 5.1. Let  $T \supseteq \text{LO}$  be a  $\Sigma_2^0 \{<\}$ -theory. Then,  $T$  has a  $\Delta_2^0$ -recursively saturated model  $\mathfrak{A}$  with a  $\Delta_2^0 \forall$ -basis selector. By Lemma 4.9, there is a recursive  $\mathfrak{B} \cong \mathfrak{A}$ . Thus,  $\mathfrak{B}$  is a recursive,  $\Delta_2^0$ -recursively saturated model of  $T$ . Corollary 2.4(2) then shows that  $\text{LO}$  is 2-thin.  $\dashv$

COROLLARY 5.3. *Every recursively axiomatizable theory of linear order has a recursive,  $\Delta_2^0$ -recursively saturated model.*  $\dashv$

This section ends with a problem. According to Ershov [6], an  $\mathcal{L}$ -theory  $T$  is  $\forall$ -finite if whenever  $T' \supseteq T$  is a (deductively closed)  $\mathcal{L}$ -theory, then  $T' \cap \forall$  is finitely axiomatizable. Furthermore,  $T$  is strongly  $\forall$ -finite if it is  $\forall$ -finite as an  $(\mathcal{L} \cup \{c_0, c_1, \dots, c_{k-1}\})$ -theory, where  $c_0, c_1, \dots, c_{k-1}$  are new constant symbols. Then it is proved in [6] that every recursively axiomatizable, strongly  $\forall$ -finite theory has a recursive model. The proof of Theorem 4.5 can be used to show that every recursively axiomatizable, strongly  $\forall$ -finite theory is 1-thin. Corollary 5.2 suggests that there might be an appropriate definition for a strongly  $\forall_2$ -theory so that: every recursively axiomatizable, strongly  $\forall_2$ -finite theory is 2-thin.

**PROBLEM 5.4.** Find an appropriate definition for a strongly  $\forall_n$ -theory, and then prove: If  $1 \leq n < \omega$ , then every recursively axiomatizable, strongly  $\forall_n$ -finite theory is  $n$ -thin.

**§6. Some more 1-thin theories.** In this section we show how to get some additional classes  $\mathbf{K}$  satisfying the hypotheses of Theorem 4.1, thereby yielding additional 1-thin theories.

Let  $\mathcal{L}$  be a finite, binary relational language; that is  $\mathcal{L} = \{R_0, R_1, R_2, \dots, R_{k-1}\}$ , where each  $R_i$  is a binary relation symbol. Say that an  $\mathcal{L}$ -structure  $\mathfrak{A} = (A, R_0, R_1, R_2, \dots, R_{k-1})$  is *irreflexive* if each  $R_i$  is irreflexive – that is, for no  $a \in A$  is  $\langle a, a \rangle \in R_i$ . For example, all posets, graphs and tournaments are irreflexive with  $k = 1$ . If  $\mathfrak{A} = (A, R_0, R_1, R_2, \dots, R_{k-1})$  is irreflexive, then a subset  $I \subseteq A$  is an *interval* if whenever  $x, y \in I$ ,  $z \in A \setminus I$  and  $i < k$ , then  $xR_i z \Leftrightarrow yR_i z$  and  $zR_i x \Leftrightarrow zR_i y$ . Obviously,  $A$ ,  $\emptyset$  and all singletons are intervals; these are the *trivial* intervals, all others being *nontrivial*. A finite irreflexive  $\mathcal{L}$ -structure  $\mathfrak{A}$  is *indecomposable* if  $|A| > 1$  and  $\mathfrak{A}$  has no nontrivial intervals. For example, every 2-element  $\mathfrak{A}$  is indecomposable; these are the *trivially* indecomposable structures.

There are no nontrivial indecomposable trees; in fact, the 4-element N-poset



is a subposet of every nontrivially indecomposable poset. (See [38] for this and much more on indecomposable structures.) For any class  $\mathbf{K}$  of finite, irreflexive  $\mathcal{L}$ -structures, we let  $\mathbf{I}(\mathbf{K})$  be the class of all nontrivially indecomposable structures in  $\mathbf{K}$ . A poset is *series-parallel* if it does not embed the N-poset. Thus, if  $\mathbf{K}$  is the class of finite series-parallel posets, then  $\mathbf{I}(\mathbf{K}) = \emptyset$ .

If  $\mathfrak{A}, \mathfrak{B}$  are finite, irreflexive  $\mathcal{L}$ -structures, then a function  $f : \mathfrak{A} \rightarrow \mathfrak{B}$  is a *homomorphism* if whenever  $R \in \mathcal{L}$  and  $a, b \in A$  are such that  $f(a) \neq f(b)$ , then  $\mathfrak{A} \models R(a, b)$  iff  $\mathfrak{B} \models R(f(a), f(b))$ . If  $\mathfrak{B} \subseteq \mathfrak{A}$  and  $f : \mathfrak{A} \rightarrow \mathfrak{B}$  is a homomorphism such that  $f(x) = x$  for all  $x \in B$ , then  $f$  is a *retraction* and  $\mathfrak{B}$  is a retract of  $\mathfrak{A}$ .

LEMMA 6.1. *Suppose  $\mathfrak{A}$  is a finite, irreflexive  $\mathcal{L}$ -structure having at least 2 elements. Then  $\mathfrak{A}$  has an indecomposable retract  $\mathfrak{B}$ .*

PROOF. The proof is by induction on the cardinality of  $\mathfrak{A}$ . If  $\mathfrak{A}$  has exactly 2 elements, then  $\mathfrak{A}$  is an indecomposable retract of itself. Now suppose  $\mathfrak{A}$  has more than 2 elements. If  $\mathfrak{A}$  is indecomposable, then  $\mathfrak{A}$  is an indecomposable retract of itself, so suppose that  $\mathfrak{A}$  is not indecomposable. Let  $I \subseteq A$  be a nontrivial interval, and let  $b \in I$ . Then let  $\mathfrak{A}' \subseteq \mathfrak{A}$  be such that  $A' = \{b\} \cup (A \setminus I)$ , and let  $f' : A \rightarrow A'$  be such that  $f'(x) = b$  if  $x \in I$  and  $f(x) = x$  otherwise. Notice that  $2 \leq |A'| < |A|$ . Then  $f' : \mathfrak{A} \rightarrow \mathfrak{A}'$  is a retraction onto  $\mathfrak{A}'$ . If  $\mathfrak{A}'$  is indecomposable, then let  $\mathfrak{B} = \mathfrak{A}'$  and we are done. Otherwise, by the inductive hypothesis, let  $g : \mathfrak{A}' \rightarrow \mathfrak{B}$  be a retraction onto an indecomposable  $\mathfrak{B} \subseteq \mathfrak{A}'$ . Then  $f = gf'$  is a retraction of  $\mathfrak{A}$  onto the retract  $\mathfrak{B}$ .  $\dashv$

We will make use of a notion even stronger than very wqo. Fix a class  $\mathbf{K}$  of structures. Suppose that  $(Q, \trianglelefteq)$  is a wqo set. A  $Q$ -labeled structure is a pair  $(\mathfrak{A}, \lambda)$ , where  $\mathfrak{A} \in \mathbf{K}$  and  $\lambda : A \rightarrow Q$ . If  $(\mathfrak{A}_1, \lambda_1), (\mathfrak{A}_2, \lambda_2)$  are  $Q$ -labeled structures, then an embedding of  $(\mathfrak{A}_1, \lambda_1)$  into  $(\mathfrak{A}_2, \lambda_2)$  is an embedding  $f : \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$  such that  $a \trianglelefteq f(a)$  for all  $a \in A_1$ . The class of  $Q$ -labeled structures will be considered to be quasi-ordered by embeddability. Then, we say that  $\mathbf{K}$  is *wqo with labels* if, for each wqo  $(Q, \trianglelefteq)$ , the class of  $Q$ -labeled structures is wqo.

If  $\mathbf{K}$  is wqo with labels, then clearly it is very wqo. It is easy to see that if  $\mathbf{K}$  is finite, then it is wqo with labels. In the following theorem, we consider  $\mathbf{K}$  for which  $\mathbf{I}(\mathbf{K})$  is wqo with labels. Thus, this is the case if  $\mathbf{I}(\mathbf{K})$  is finite. The proof of the following theorem is an extension of the proof in [35] of its special case where  $\mathbf{K}$  is the class of finite series-parallel posets. It is closely patterned after Nash-Williams' elegant proof [30] of Kruskal's Theorem. See [27] for a good exposition.

THEOREM 6.2. *Suppose that  $\mathbf{K}$  is a hereditary class of finite irreflexive, binary relational structures. If  $\mathbf{I}(\mathbf{K})$  is wqo with labels, then  $\mathbf{K}$  is wqo with labels.*

PROOF. This proof uses the method of minimal bad sequences, which Nash-Williams [30] originated.

Let  $(Q, \trianglelefteq)$  be a wqo set. Let  $\mathbf{K}'$  be the class of  $Q$ -labeled structures in  $\mathbf{K}$  quasi-ordered by embeddability. Suppose that  $\mathbf{K}'$  is not wqo, and let  $(\mathfrak{A}_0, \lambda_0), (\mathfrak{A}_1, \lambda_1), (\mathfrak{A}_2, \lambda_2), \dots$  be a minimal bad sequence from  $\mathbf{K}'$ . (Explanation: this sequence is *bad* if whenever  $i < j < \omega$ , then  $(\mathfrak{A}_i, \lambda_i)$  is not embeddable in  $(\mathfrak{A}_j, \lambda_j)$ , and this bad sequence is *minimal* if, whenever  $(\mathfrak{B}_0, \mu_0), (\mathfrak{B}_1, \mu_1), (\mathfrak{B}_2, \mu_2), \dots$  is another bad sequence from  $\mathbf{K}'$  and  $i < \omega$  is the least such that  $|A_i| \neq |B_i|$ , then  $|A_i| < |B_i|$ .)

Using Lemma 6.1, for each  $i < \omega$ , let  $f_i : \mathfrak{A}_i \rightarrow \mathfrak{B}_i \subseteq \mathfrak{A}_i$  be a retraction onto  $\mathfrak{B}_i$  such that whenever  $|A_i| \geq 2$ , then  $\mathfrak{B}_i$  is indecomposable. If  $i < \omega$

and  $x \in B_i$ , let  $\mathfrak{A}'_{i,x}$  be the  $\mathcal{Q}$ -labeled structure  $(\mathfrak{A}_i \upharpoonright f_i^{-1}(x), \lambda_i \upharpoonright f_i^{-1}(x))$ . Let  $L = \{\mathfrak{A}_{i,x} : i < \omega, x \in B_i\} \subseteq K'$ .

CLAIM.  $L$  is wqo.

For, suppose  $L$  is not wqo. Then there are  $i_0 < i_1 < i_2 < \dots$  and  $x_j \in B_{i_j}$  for each  $j < \omega$  such that  $|B_{i_j}| \geq 2$  and  $\mathfrak{A}'_{i_j, x_j}$  is not embeddable in  $\mathfrak{A}'_{i_k, x_k}$  whenever  $j < k < \omega$ . Then the sequence

$$(\mathfrak{A}_0, \lambda_0), (\mathfrak{A}_1, \lambda_1), \dots, (\mathfrak{A}_{i_0-1}, \lambda_{i_0-1}), \mathfrak{A}'_{i_0, x_0}, \mathfrak{A}'_{i_1, x_1}, \mathfrak{A}'_{i_2, x_2}, \dots$$

is bad and contradicts the minimality of  $(\mathfrak{A}_0, \lambda_0), (\mathfrak{A}_1, \lambda_1), (\mathfrak{A}_2, \lambda_2), \dots$ .

For  $i < \omega$ , let  $\mu_i$  be the  $L$ -labeling of  $\mathfrak{B}_i$  where  $\mu_i(x) = \mathfrak{A}'_{i,x}$ . Since  $I(K)$  is wqo with labels, there are  $i < j < \omega$  such that  $(\mathfrak{B}_i, \mu_i)$  is embeddable in  $(\mathfrak{B}_j, \mu_j)$ . Clearly, this implies that  $(\mathfrak{A}_i, \lambda_i)$  is embeddable in  $(\mathfrak{A}_j, \lambda_j)$ , which is a contradiction.  $\neg$

One consequence of Theorem 6.2 is that Corollary 4.3 would remain true if both occurrences of the word “tree” were replaced with “series-parallel poset”.

I have a favorite application of Theorem 6.2. A nontrivially indecomposable  $\mathcal{L}$ -structure  $\mathfrak{A}$  is defined in [38] to be *critically* indecomposable if  $|A| > 2$  and it has no indecomposable substructure  $\mathfrak{B} \subseteq \mathfrak{A}$  such that  $|B| = |A| - 1$ . It can be seen from [38] that if  $K$  is any class of finite irreflexive, binary relational structures, then the subclass of all critically indecomposable structures in  $K$  is wqo, and with the help of Higman’s Theorem, it can be seen to be wqo with labels. This is so despite the fact that if  $K$  is the class of all finite graphs, posets or tournaments, then there are infinitely many pairwise nonisomorphic critically indecomposable structures in  $K$ .

Let us say that an indecomposable structure that is not critically indecomposable is *noncritically* indecomposable, and then let  $I_0(K)$  be the class of those noncritically indecomposable structures in  $I(K)$ .

COROLLARY 6.3. *Let  $\mathcal{L}$  be a finite, binary relational language, and let  $K$  be a hereditary class of finite, irreflexive  $\mathcal{L}$ -structures. If  $I_0(K)$  is finite, then  $K$  is wqo with labels.*  $\neg$

Suppose that  $I$  is a class of finite, nontrivially indecomposable  $\mathcal{L}$ -structures. Then, let  $K$  be the class of finite, irreflexive  $\mathcal{L}$  structures  $\mathfrak{A}$  all of whose nontrivially indecomposable substructures are in  $I$ . In other words,  $K$  is the largest hereditary class of finite, irreflexive  $\mathcal{L}$ -structures such that  $I(K) = I$ . If  $I$  has only finitely many noncritically indecomposable structures, then Corollary 6.3 applies to  $K$ . Such classes  $K$  seem to be especially well behaved. Many of the results and proofs from [36] can be extended to the theory  $T = T_K = \forall \cap \text{Th}(K)$ . For one example,  $T$  is decidable; in fact, the monadic second-order theory of its countable models is decidable. Another example: every  $\aleph_0$ -completion of  $T$  is decidable.

As part of a program [20] of studying wqo classes of tournaments, Latka [21] has considered a class of finite tournaments to which Corollary 6.3 applies. Let  $N_5$  be the tournament on the set  $\{0, 1, 2, 3, 4\}$  in which  $i \rightarrow j$  iff  $j - i \geq 2$  or  $i - j = 1$ . We say that a tournament is  $N_5$ -free if it does not embed  $N_5$ . If  $\mathbf{K}$  is the class of finite  $N_5$ -free tournaments, then Latka [21] proves that there are exactly two tournaments in  $\mathbf{I}_0(\mathbf{K})$  (even though  $\mathbf{I}(\mathbf{K})$  is infinite) and, therefore, that this class is wqo. In fact, it is wqo with labels, thereby implying the following corollary whose proof is just like the proof of Corollary 4.3.

**COROLLARY 6.4.** *Suppose  $\varphi(x, y)$  is a formula such that there is a model  $\mathcal{M} \models \text{PA}$  for which  $\mathcal{M} \upharpoonright [\varphi]$  is an  $N_5$ -free tournament. Then there is a recursively saturated model  $\mathcal{M} \models \text{PA}$  for which  $\mathcal{M} \upharpoonright [\varphi]$  is a recursive  $N_5$ -free tournament.*  $\dashv$

Letting  $T_5$  be the theory of the class of  $N_5$ -free tournaments, we see from Theorem 4.1 that  $T_5$  is 1-thin and, therefore, is not 1-rich by Lemma 2.5. This suggests the problem of determining exactly how thin  $T_5$  is and how much it fails to be rich.

- QUESTION 6.5.** (a) Is  $T_5$  2-rich?  
 (b) 2-thin?  
 (c) Is there a  $\Delta_2^0$  completion of  $T_5$  that has no recursive model?

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# HISTORY OF CONSTRUCTIVISM IN THE 20TH CENTURY

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**§1. Introduction.** In this survey of the history of constructivism,<sup>1</sup> more space has been devoted to early developments (up till ca. 1965) than to the work of the later decades. Not only because most of the concepts and general insights have emerged before 1965, but also for practical reasons: much of the work since 1965 is of a too technical and complicated nature to be described adequately within the limits of this article.

Constructivism is a point of view (or an attitude) concerning the methods and objects of mathematics which is normative: not only does it interpret existing mathematics according to certain principles, but it also rejects methods and results not conforming to such principles as unfounded or speculative (the rejection is not always absolute, but sometimes only a matter of degree: a decided preference for constructive concepts and methods). In this sense the various forms of constructivism are all ‘ideological’ in character.

Constructivism as a specific viewpoint emerges in the final quarter of the 19th century, and may be regarded as a reaction to the rapidly increasing use of highly abstract concepts and methods of proof in mathematics, a trend exemplified by the works of R. Dedekind and G. Cantor.

The mathematics before the last quarter of the 19th century is, from the viewpoint of today, in the main constructive, with the notable exception of geometry, where proof by contradiction was commonly accepted and widely employed.

Characteristic for the constructivist trend is the insistence that mathematical objects are to be constructed (mental constructions) or computed; thus theorems asserting the existence of certain objects should by their proofs give us the means of constructing objects whose existence is being asserted.

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<sup>1</sup>This paper appeared as preprint *History of Constructivism in the Twentieth Century* in 1991 in the ITLI Prepublication Series ML-91-05. Although it was intended that the paper should be translated and appear in an Italian encyclopedia, it appeared only in 2006 in a severely curtailed and mutilated form, as a short chapter ‘L’ntuizionismo di Brouwer’ in volume 8 (*Storia della Scienza*), pp. 121–124, of *La Seconda Rivoluzione Scientifica*, published by the Istituto della Enciclopedia Italiana, Roma. The original version describes the history of constructivism in mathematics for the period 1900–1990, from the viewpoint of 1991.

L. Kronecker may be described as the first conscious constructivist. For Kronecker, only the natural numbers were ‘God-given’; all other mathematical objects ought to be explained in terms of natural numbers (at least in algebra). Assertions of existence should be backed up by constructions, and the properties of numbers defined in mathematics should be decidable in finitely many steps. This is in the spirit of finitism (see the next section). Kronecker strongly opposed Cantor’s set theory, but in his publications he demonstrated his attitude more by example than by an explicit discussion of his tenets.

The principal constructivistic trends in this century are finitism already mentioned (Section 2), semi-intuitionism and predicativism (Section 3), intuitionism (Section 4), Markov’s constructivism (Section 7) and Bishop’s constructivism (Section 8).

Constructivism, in particular intuitionism, has given rise to a considerable amount of metamathematical research (Sections 5, 6).

**Notations.** We use  $N$ ,  $Q$ ,  $R$  for the set of natural, rational, and real numbers respectively.

For logical symbols we use  $\neg$  (not),  $\perp$  (falsehood),  $\rightarrow$  (implication),  $\wedge$  (conjunction, and),  $\vee$  (disjunction, or),  $\forall$  (for all),  $\exists$  (there exists).  $A, B, C$  are arbitrary propositions or formulas of the language under discussion.

Usually  $n, m, k$  are elements of  $N$ ,  $\alpha, \beta$  number-theoretic functions (infinite sequences of natural numbers);  $\bar{\alpha}n \equiv \langle \alpha 0, \alpha 1, \dots, \alpha(n-1) \rangle$  is the initial segment of  $\alpha$  of length  $n$ ;  $\langle \rangle$  is the empty sequence.  $n * m$  is the concatenation of  $n$  and  $m$ .

*IPC*, *IQC*, *HA* are systems of intuitionistic propositional logic, predicate logic, and arithmetic respectively. In arithmetic,  $S$  is the successor function (i.e.  $Sx = x + 1$ ). These intuitionistic systems differ from standard axiomatizations of the corresponding classical systems only by the absence of the *principle PEM of the excluded middle*

$$\text{PEM} \quad A \vee \neg A,$$

or the principle of double negation  $\neg\neg A \rightarrow A$ .

## §2. Finitism.

**2.1. Finitist mathematics.** Finitism may be characterized as based on the concept of natural number (or finite, concretely representable structure), which is taken to entail the acceptance of proof by induction and definition by recursion.

Abstract notions, such as ‘constructive proof’, ‘arbitrary number-theoretic function’ are rejected. Statements involving quantifiers are finitistically interpreted in terms of quantifier-free statements. Thus an existential statement

$\exists x Ax$  is regarded as a partial communication, to be supplemented by providing an  $x$  which satisfies  $A$ . Establishing  $\neg\forall x Ax$  finitistically means: providing a particular  $x$  such that  $Ax$  is false.

In this century, T. Skolem was the first to contribute substantially to finitist mathematics; he showed that a fair part of arithmetic could be developed in a calculus without bound variables, and with induction over quantifier-free expressions only. Introduction of functions by primitive recursion is freely allowed.<sup>2</sup> Skolem does not present his results in a formal context, nor does he try to delimit precisely the extent of finitist reasoning.

Since the idea of finitist reasoning was an essential ingredient in giving a precise formulation of Hilbert's programme (the consistency proof for mathematics should use intuitively justified means, namely finitist reasoning), Skolem's work is extensively reported by D. Hilbert and P. Bernays. Hilbert also attempted to circumscribe the extent of finitist methods in a global way; the final result is found in 'Die Grundlagen der Mathematik'.<sup>3</sup>

In 1941 H. B. Curry and R. L. Goodstein independently formulated a purely equational calculus **PRA** for primitive recursive arithmetic in which Skolem's arguments could be formalized, and showed that the addition of classical propositional logic to **PRA** is conservative (i.e. no new equations become provable). **PRA** contains symbols for all primitive recursive functions, with their defining equations, and an induction rule of the form: if  $t[0] = t'[0]$ , and  $t[Sx] = s[x, t[x]]$ ,  $t'[Sx] = s[x, t'[x]]$  has been derived, then we may conclude  $t[x] = t'[x]$ .

Goodstein carried the development of finitist arithmetic beyond Skolem's results, and also showed how to treat parts of analysis by finitist means.<sup>4</sup>

W. W. Tait in [Tait 1981] attempts to delimit the scope of finitist reasoning. He defends the thesis that **PRA** is indeed the limit of finitist reasoning. Any finitely axiomatized part of **PRA** can be recognized as finitist, but never all of **PRA**, since this would require us to accept the *general* notion of a primitive recursive function, which is not finitist.

In subsequent years there has been a lot of metamathematical work showing that large parts of mathematics have an indirect finitist justification, namely by results of the form: a weak system **S** in a language with strong expressive power is shown to be consistent by methods formalizable in **PRA**, from which it may be concluded that **S** is conservative over **PRA**. A survey of such results is given in S. Simpson's [Simpson 1988].

**2.2. Actualism.** A remark made in various forms by many authors, from G. Mannoury in 1909 onwards, is the observation that already the natural

<sup>2</sup>See [Skolem 1923].

<sup>3</sup>[Hilbert and Bernays 1934], chapter 2.

<sup>4</sup>[Goodstein 1957, 1959].

number concept involves a strong idealization of the idea of ‘concretely representable’ or ‘visualizable’. Such an idealization is implicit in the assumption that all natural numbers are constructions of the same kind, whether we talk about very small numbers such as 3 or 5, or extremely large ones such as  $9^9$ . In reality, we cannot handle  $9^9$  without some understanding of the general concept of exponentiation. The objection to finitism, that it is not restricted to objects which can be *actually* realized (physically, or in our imagination) one might call the ‘actualist critique’, and a programme taking the actualist critique into account, *actualism* (sometimes called ‘ultra-intuitionism’, or ‘ultra-finitism’).

The first author to defend an actualist programme, was A. S. Esenin-Vol’pin in 1957. He intended to give a consistency proof for **ZF** using only ‘ultra-intuitionist’ means. Up till now the development of ‘actualist’ mathematics has not made much progress — there appear to be inherent difficulties associated with an actualist programme.

However, mention should be made of Parikh’s [Parikh 1971], motivated by the actualist criticism of finitism, where is indicated, by technical results, the considerable difference in character between addition and multiplication on the one hand and exponentiation on the other hand. Together with work in complexity theory, this paper stimulated the research on polynomially bounded arithmetic, as an example of which may be quoted S. Buss’s monograph *Bounded Arithmetic*.<sup>5</sup>

### §3. Predicativism and semi-intuitionism.

**3.1. Poincaré.** The French mathematician H. Poincaré wrote many essays on the philosophy of mathematics and the sciences, collected in [Poincaré 1902, 1905, 1908, 1913]; his ideas played an important role in the debate on the foundations of mathematics in the early part of this century. One cannot extract a unified and coherent point of view from Poincaré’s writings. On the one hand he is a forerunner of the (semi-) intuitionists and predicativists, on the other hand he sometimes expresses formalist views, namely where he states that existence in mathematics can never mean anything but freedom from contradiction.

For the history of constructivism, Poincaré is important for two reasons:

(1) Explicit discussion and emphasis on the role of intuition in mathematics, more especially ‘the intuition of pure number’. This intuition gives us the principle of induction for the natural numbers, characterized by Poincaré as a ‘synthetic judgment a priori’. That is to say, the principle is neither tautological (i.e. justified by pure logic), nor is it derived from experience; instead, it is a consequence of our intuitive understanding of the notion of number. In this respect, Poincaré agrees with the semi-intuitionists and Brouwer.

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<sup>5</sup>[Buss 1986].

(2) According to Poincaré, the set-theoretic paradoxes are due to a vicious circle, namely the admission of *impredicative* definitions: defining an object  $N$  by referring to a totality  $E$  (in particular by quantifying over  $E$ ), while  $N$  itself belongs to  $E$ .

Poincaré's standard example is J. Richard's paradox:<sup>6</sup> let  $E$  be the totality of all real numbers given by infinite decimal fractions, definable in finitely many words.  $E$  is clearly countable, so by a wellknown Cantor style diagonal argument we can define a real  $N$  not in  $E$ . But  $N$  has been defined in finitely many words! However, Poincaré, adopting Richard's conclusion, points out that the definition of  $N$  as element of  $E$  refers to  $E$  itself and is therefore impredicative. For a detailed discussion of Poincaré's philosophy of mathematics, see [Mooij 1966].

**3.2. The semi-intuitionists.** The term 'semi-intuitionists' or 'empirists' refers to a group of French mathematicians, in particular E. Borel, H. Lebesgue, R. Baire, and the Russian mathematician N. N. Luzin. Their discussions of foundational problems are always in direct connection with specific mathematical developments, and thus have an 'ad hoc', local, character; also the views within the group differ.

What the semi-intuitionists have in common is the idea that, even if mathematical objects exist independently of the human mind, mathematics can only deal with such objects if we can also attain them by mentally constructing them, i.e. have access to them by our intuition; in practice, this means that they should be explicitly definable. In addition, pragmatic considerations occur: one is not interested in arbitrary objects of a certain kind, but only in the ones which play an important role in mathematics (which 'occur in nature' in a manner of speaking).

We shall illustrate semi-intuitionism by a summary of the views of Borel, the most explicit and outspoken of the semi-intuitionists. According to Borel, one can assert the existence of an object only if one can define it unambiguously in finitely many words. In this concern with definability there is a link with predicativism (as described below in 3.4); on the other hand, there is in Borel's writings no explicit concern with impredicativity and the vicious circle principle, as in the writings of Poincaré and H. Weyl. (The Borel sets, introduced by Borel in the development of measure theory, are a 'continuous' analogue of the hyperarithmetic sets, which may be regarded as a first approximation to the general notion of a predicative subset of the natural numbers, but this was not known to Borel.)

With respect to the reality of the countably infinite, Borel takes a somewhat pragmatic attitude: while conceding the strength of the position of the strict finitist, he observes that the countably infinite plays a very essential role in

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<sup>6</sup>[Richard 1905].

mathematics, and mathematicians have in practice always agreed on the correct use of the notion:

The notion of the countably infinite appears therefore [ . . . ] as a limit possibility conceived by our imagination, just like the ideal line or the perfect circle.<sup>7</sup>

In other words, the natural numbers and the principle of complete induction are intuitively clear.

Borel explains the Richard paradox, not by referring to the impredicative character of the definition of  $N$ , but by the observation that  $N$  has not been unambiguously defined; the collection  $E$  is countable, but not effectively enumerable and hence the construction of  $N$  cannot be carried out.<sup>8</sup> This distinction would later obtain a precise formulation in recursion theory.

Borel explicitly introduced the notion of a calculable real number; a function  $f$  is calculable if  $f(x)$  is calculable for each calculable  $x$ , and he observes that a calculable function must necessarily be continuous, at least for calculable arguments<sup>9</sup> This foreshadows Brouwer's well-known theorem on the continuity of all real functions<sup>10</sup>.

**3.3. Borel and the continuum.** Borel rejected the general notion of an uncountable set, as an essentially negative notion. According to Borel, Cantor's diagonal argument only shows that the continuum cannot be exhausted by a countable set. Obviously, on this view the continuum presents a problem. On the one hand it is the basic concept in analysis, on the other hand it cannot be described as 'the set of its (definable) elements' since it is uncountable. Thus Borel remarks in 1908:

[ . . . ] the notion of the continuum, which is the only well-known example of a non-denumerable set, that is to say a set of which the mathematicians have a clear idea in common (or believe to have in common, which in practice amounts to the same thing). I regard that notion as acquired from the geometric intuition of the continuum; it is well-known that the complete arithmetical notion of the continuum requires that one admits the legitimacy of an infinity of countably many successive arbitrary choices.<sup>11</sup>

(The term 'arithmetic theory of the continuum', or 'arithmetization of the continuum', frequently appears in discussions of constructivism in the early part of this century. By this is meant the characterization of the continuum as a *set* of reals, where the real numbers are obtained as equivalence classes of fundamental sequences of rationals, or as Dedekind cuts. Since the rationals

<sup>7</sup>[Borel 1914], p. 179.

<sup>8</sup>[Borel 1914], p. 165.

<sup>9</sup>[Borel 1914], p. 223.

<sup>10</sup>Cf. Section 6.1.

<sup>11</sup>[Borel 1914], p. 162.

can be enumerated, this achieves a reduction of the theory of the continuum to number-theoretical or ‘arithmetical’ functions.)

In 1912 Borel remarks that one can reason about certain classes of objects, such as the reals, since the class is defined in finitely many words, even if not all elements of the class are finitely definable. Therefore Borel had to accept the continuum as a primitive concept, not reducible to an arithmetical theory of the continuum.

Although the fact is not mentioned by Borel, the idea of a continuum consisting of only countably many definable reals suggests a ‘measure-theoretic paradox’, for if the reals in  $[0, 1]$  are countable, one can give a covering by a sequence of intervals  $I_0, I_1, I_2, \dots$  with  $\sum_{n=0}^{\infty} |I_n| < 1$ , where  $|I_n|$  is the length of  $I_n$ . (Such coverings are called *singular*.) The paradox is repeatedly mentioned in Brouwer’s publications (e.g. [Brouwer 1930]), as a proof of the superiority of his theory of the continuum.

**3.4. Weyl.** Motivated by his rejection of the platonistic view of mathematics prevalent in Cantorian set theory and Dedekind’s foundation of the natural number concept, Weyl, in his short monograph ‘Das Kontinuum’<sup>12</sup> formulated a programme for predicative mathematics; it appears that Weyl had arrived at his position independently of Poincaré and B. Russell. Predicativism may be characterized as ‘constructivism’ with respect to definitions of sets (but not with respect to the use of logic): sets are constructed from below, not characterized by singling them out among the members of a totality conceived as previously existing.

Weyl accepted classical logic and the set of natural numbers with induction and definition by recursion as unproblematic. Since the totality of natural numbers is accepted, all arithmetical predicates make sense as sets and we can quantify over them. The arithmetically definable sets are the *sets of rank 1*, the first level of a predicative hierarchy of ranked sets; sets of higher rank are obtained by permitting quantification over sets of lower rank in their definition. Weyl intended to keep the developments simple by restricting attention to sets of rank 1.

On the basis of these principles Weyl was able to show for example: Cauchy sequences of real numbers have a limit; every bounded monotone sequence of reals has a limit; every countable covering by open intervals of a bounded closed interval has a finite subcovering; the intermediate value theorem holds (i.e. a function changing sign on an interval has a zero in the interval); a continuous function on a bounded closed interval has a maximum and a minimum.

After his monograph appeared, Weyl became for a short period converted to Brouwer’s intuitionism. Later he took a more detached view, refusing the exclusive adoption of either a constructive or an abstract axiomatic approach.

<sup>12</sup>[Weyl 1918].

Although Weyl retained a lifelong interest in the foundations of mathematics, he did not influence the developments after 1918. For an excellent summary of Weyl's development, as well as a technical analysis of 'Das Kontinuum' see S. Feferman's 'Weyl Vindicated'.<sup>13</sup>

**3.5. Predicativism after 'das Kontinuum'.** After Weyl's monograph predicativism rested until the late fifties, when interest revived in the work of M. Kondô, A. Grzegorczyk and G. Kreisel. Kreisel showed that the so-called hyperarithmetic sets known from recursion theory constituted an upper bound for the notion 'predicatively definable set of natural numbers'. Feferman and K. Schütte addressed the question of the precise extent of predicative analysis; they managed to give a characterization of its proof theoretic ordinal. Type free formalizations for predicative analysis with sets of all (predicative) ranks were developed. In recent years also many formalisms have been shown to be indirectly reducible to predicative systems, cf. [Feferman 1988a].

P. Lorenzen's [1955, 1965] may be regarded as a direct continuation of Weyl's programme.

#### §4. Brouwerian intuitionism.

**4.1. Early period.** In his thesis 'Over de Grondslagen der Wiskunde'<sup>14</sup> the Dutch mathematician L. E. J. Brouwer defended, more radically and more consistently than the semi-intuitionists, an intuitionist conception of mathematics. Brouwer's philosophy of mathematics is embedded in a general philosophy, the essentials of which are found already in [Brouwer 1905]. To these philosophical views Brouwer adhered all his life; a late statement may be found in [Brouwer 1949]. With respect to mathematics, Brouwer's main ideas are:

1. Mathematics is not formal; the objects of mathematics are mental constructions in the mind of the (ideal) mathematician. Only the thought constructions of the (idealized) mathematician are exact.
2. Mathematics is independent of experience in the outside world, and mathematics is in principle also independent of language. Communication by language may serve to suggest similar thought constructions to others, but there is no guarantee that these other constructions are the same. (This is a solipsistic element in Brouwer's philosophy.)
3. Mathematics does not depend on logic; on the contrary, logic is part of mathematics.

These principles led to a programme of reconstruction of mathematics on intuitionistic principles ('Brouwer's programme' or 'BP' for short). During the early period, from say 1907 until 1913, Brouwer did the major part of his

<sup>13</sup>[Feferman 1988b].

<sup>14</sup>[Brouwer 1907].

work in (classical) topology and contributed little to BP. In these years his view of the continuum and of countable sets is quite similar to Borel's position on these matters. Thus he writes:

The continuum as a whole was intuitively given to us; a construction of the continuum, an act which would create "all" its parts as individualized by the mathematical intuition is unthinkable and impossible. The mathematical intuition is not capable of creating other than countable quantities in an individualized way.<sup>15</sup>

On the other hand, already in this early period there are also clear differences; thus Brouwer did not follow Borel in his pragmatic intersubjectivism, and tries to explain the natural numbers and the continuum as two aspects of a single intuition ('the primeval intuition').

Another important difference with Borel c.s. is, that Brouwer soon after finishing his thesis realized that classical logic did not apply to his mathematics (see the next section). Nevertheless, until circa 1913 Brouwer held some of the views of the semi-intuitionists and did not publicly dissociate himself from them.

**4.2. Weak counterexamples and the creative subject.** Already in [Brouwer 1908] a typically intuitionistic kind of counterexample to certain statements  $A$  of classical mathematics was introduced, not counterexamples in the strict sense of deriving a contradiction from the statement  $A$ , but examples showing that, assuming that we can prove  $A$  intuitionistically, it would follow that we had a solution to a problem known to be as yet unsolved. Undue attention given to these examples often created for outsiders the erroneous impression that intuitionism was mainly a negative activity of criticizing classical mathematics.

For example, consider the set  $X \equiv \{x : x = 1 \vee (x = 2 \wedge F)\}$  where  $F$  is any as yet undecided mathematical statement, such as the Riemann hypothesis.  $X$  is a subset of the finite set  $\{1, 2\}$ , but we cannot prove  $X$  to be finite, since this would require us to decide whether  $X$  has only one, or two elements, and for this we would have to decide  $F$ . (Intuitionistically, a set is finite if it can be brought into a constructively specified 1-1 correspondence with an initial part of the natural numbers.) So we have found a weak counterexample to the statement 'a subset of a finite set is finite'.

By choosing our undecided problems suitably, it is also possible to give a weak counterexample to: 'for all reals  $x$ ,  $x < 0$  or  $x = 0$  or  $x > 0$ ', or to 'for all reals  $x$ ,  $x \leq 0$  or  $x \geq 0$ '. Brouwer used these examples to show the need for an intuitionistic revision of certain parts of the classical theory, and to demonstrate how classically equivalent definitions corresponded to distinct intuitionistic notions.

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<sup>15</sup>[Brouwer 1907], p. 62, cf. also p. 10.

G. F. C. Griss, in a number of publications between 1944 and 1955, advocated a form of intuitionistic mathematics without negation, since one cannot have a clear conception of what it means to give proof of a proposition which turns out to be contradictory (cf. the interpretation of intuitionistic implication and negation in Subsection 5.2).

In reaction to Griss, Brouwer published in 1948 an example of a statement in the form of a negation which could not be replaced by a positive statement. Brouwer's example involved a real number defined by explicit reference to stages in the activity of the ideal mathematician trying to prove an as yet undecided statement  $A$ . An essential ingredient of Brouwer's argument is that if the ideal mathematician (*creative subject* in Brouwer's terminology) is certain that at no future stage of his mathematical activity he will find  $A$  to be true, it means that he knows that  $\neg A$  (Brouwer seems to have used such examples in lectures since 1927).

The argument illustrates the extreme solipsistic consequences of Brouwer's intuitionism. Because of their philosophical impact, these examples generated a good deal of discussion and inspired some metamathematical research, but their impact on BP was very limited, in fact almost nil.

**4.3. Brouwer's programme.** After 1912, the year in which he obtained a professorship at the University of Amsterdam, Brouwer started in earnest on his programme, and soon discovered that for a fruitful reconstruction of analysis his ideas on the continuum needed revision. Around 1913 he must have realized that the notion of choice sequence (appearing in a rather different setting in Borel's discussion of the axiom of choice) could be legitimized from his viewpoint and offered all the advantages of an 'arithmetical' theory of the continuum. The first paper where the notion is actually used is Brouwer's [Brouwer 1919].

In the period up to 1928 he reconstructed parts of the theory of pointsets, some function theory, developed a theory of countable well-orderings and, together with his student B. de Loor, gave an intuitionistic proof of the fundamental theorem of algebra.

Brouwer's ideas became more widely known only after 1920, when he lectured on them at many places, especially in Germany.

After 1928, Brouwer displayed very little mathematical activity, presumably as a result of a conflict in the board of editors of the 'Mathematische Annalen'<sup>16</sup>. From 1923 onwards, M. J. Belinfante and A. Heyting, and later also Heyting's students continued BP. Belinfante investigated intuitionistic complex function theory in the thirties. Heyting dealt with intuitionistic projective geometry and algebra (in particular linear algebra and elimination theory). In the period 1952–1967, six of Heyting's Ph.D students wrote theses on subjects such as intuitionistic topology, measure theory, theory of Hilbert spaces, the

<sup>16</sup>See [Van Dalen 1990].

Radon integral, intuitionistic affine geometry. After 1974, interesting contributions were made by W. Veldman, who studied the intuitionistic analytic hierarchy.

The discovery of a precise notion of algorithm in the thirties (the notion of a (general) recursive function) as a result of the work of A. Church, K. Gödel, J. Herbrand, A. M. Turing and S. C. Kleene, did not affect intuitionism. This is not really surprising: most of these characterizations describe algorithms by specifying a narrow language in which they can be expressed, which is utterly alien to Brouwer's view of mathematics as the languageless activity of the ideal mathematician. Turing's analysis is not tied to a specific formalism. Nevertheless he bases his informal arguments for the generality of his notion of algorithm on the manipulation of symbols and appeals to physical limitations on computing; and such ideas do not fit into Brouwer's ideas on mathematics as a 'free creation'.

Without Heyting's sustained efforts to explain Brouwer's ideas and to make them more widely known, interest in intuitionism might well have died out in the thirties. However, most of the interest in intuitionism concerned its metamathematics, not BP, contrary to Heyting's intentions. Heyting's monograph [Heyting 1956] was instrumental in generating a wider interest in intuitionism.

The next two sections will be devoted to the codification of intuitionistic logic and the gradual emergence of the metamathematics of constructive theories.

## §5. Intuitionistic logic and arithmetic.

**5.1. L. E. J. Brouwer and intuitionistic logic.** The fact that Brouwer's approach to mathematics also required a revision of the principles of classical logic was not yet clearly realized by him while writing his thesis, but in 1908 Brouwer explicitly noted that intuitionism required a different logic. In particular, he noted that the principle of the excluded middle  $A \vee \neg A$  is not intuitionistically valid. Implicitly, of course, the meaning of the logical operators had been adapted to the intuitionistic context, that is the intuitionistic meaning of a statement  $A \vee \neg A$  is different from the classical one. From [Brouwer 1908]:

is it allowed, in purely mathematical constructions and transformations, to neglect for some time the idea of the mathematical system under construction and to operate in the corresponding linguistic structure, following the principles of [ . . . ] and can we have confidence that each part of the argument can be justified by recalling to the mind the corresponding mathematical construction?

A first important technical contribution to intuitionistic logic is made in ([Brouwer 1924b], *Zerlegung*), namely the observation that  $\neg\neg A \leftrightarrow A$  is an intuitionistic logical law.

**5.2. The Brouwer-Heyting-Kolmogorov interpretation.** The standard informal interpretation of logical operators in intuitionistic logic is the so-called *proof-interpretation* or *Brouwer-Heyting-Kolmogorov interpretation* (*BHK-interpretation* for short). The formalization of intuitionistic logic started before this interpretation was actually formulated, but it is preferable to discuss the BHK-interpretation first since it facilitates the understanding of the more technical results. On the BHK-interpretation, the meaning of a statement  $A$  is given by explaining what constitutes a proof of  $A$ , and *proof of  $A$*  for logically compound  $A$  is explained in terms of what it means to give a proof of its constituents. Thus, for propositional logic:

1. A proof of  $A \wedge B$  is given by presenting a proof of  $A$  and a proof of  $B$ .
2. A proof of  $A \vee B$  is given by presenting either a proof of  $A$  or a proof of  $B$ .
3. A proof of  $A \rightarrow B$  is a construction which transforms any proof of  $A$  into a proof of  $B$ .
4. Absurdity  $\perp$  ('the contradiction') has no proof; a proof of  $\neg A$  is a construction which transforms any supposed proof of  $A$  into a proof of  $\perp$ .

Such an interpretation is implicit in Brouwer's writings, e.g. [Brouwer 1908, 1924b, Zerlegung] and has been made explicit by Heyting for predicate logic in [Heyting 1934], and by A. N. Kolmogorov in [Kolmogorov 1932] for propositional logic.

Kolmogorov formulated what is essentially the same interpretation in different terms: he regarded propositions as problems, and logically compound assertions as problems explained in terms of simpler problems, e.g.  $A \rightarrow B$  represents the problem of reducing the solution of  $B$  to the solution of  $A$ . Initially Heyting and Kolmogorov regarded their respective interpretations as distinct; Kolmogorov stressed that his interpretation also makes sense in a classical setting. Later Heyting realized that, at least in an intuitionistic setting, both interpretations are practically the same.

**5.3. Formal intuitionistic logic and arithmetic through 1940.** Kolmogorov's paper from 1925, written in Russian, is the earliest published formalization of a fragment of intuitionistic logic, and represents a remarkable achievement, but had very little effect on the developments (in 1933 still unknown to Gödel, and not seen by Heyting in 1934). Kolmogorov does not assume the 'ex falso sequitur quodlibet'  $P \rightarrow (\neg P \rightarrow Q)$  which is justifiable on the basis of the BHK-interpretation. The system of intuitionistic logic with the 'ex falso' deleted became known as *minimal logic* and is of some interest in connection with completeness problems.

V. I. Glivenko presented in 1928 a (not complete) formalization of intuitionistic propositional logic and derives from this informally  $\neg\neg(\neg P \vee P)$ ,  $\neg\neg\neg P \rightarrow \neg P$ ,  $(\neg P \vee P \rightarrow \neg Q) \rightarrow \neg Q$ , and uses these theorems to show that

the interpretation of [Barzin and Errera 1927] of Brouwer's logic, according to which a proposition intuitionistically is either true, or false, or 'tierce', is untenable; a nice example of the use of formalization to settle a philosophical debate.

Heyting wrote a prize essay on the formalization of intuitionistic mathematics which was crowned by the Dutch Mathematical Association in 1928; the essay appeared in revised form in 1930. The first of the three papers [Heyting 1930] contains a formalization of intuitionistic propositional logic in its present extent. The second paper deals with predicate logic and arithmetic. Predicate logic does not yet appear in its final form, due to a defective treatment of substitution, and the (not quite consistent) germs of a theory permitting non-denoting terms. Arithmetic as presented in the second paper is a fragment of Heyting arithmetic as it is understood today, since there are axioms for addition (in the guise of a definition) but not for multiplication. The third paper deals with analysis. The system is very weak due to a lack of existence axioms for sets and functions.

In 1929 Glivenko formulated and proved as a result of his correspondence with Heyting the 'Glivenko theorem': in propositional logic  $\neg\neg A$  is intuitionistically provable if and only if  $A$  is classically provable.

Kolmogorov in his paper from 1925 describes an embedding of classical propositional logic into (his fragment of) intuitionistic propositional logic, thereby anticipating the work of Gödel in 1933 and G. Gentzen (also dating from 1933, but published only in 1965), and argues that this embedding is capable of generalization to stronger systems. Gödel's embedding is formulated for arithmetic, but can be adapted in an obvious way to predicate logic. In Gentzen's version prime formulas  $P$  are first replaced by  $\neg\neg P$ , and the operators  $\dots \vee \dots$ ,  $\exists x \dots$  by  $\neg(\neg\dots \wedge \neg\dots)$ ,  $\neg\forall x\neg\dots$  respectively. In Kolmogorov's version  $\neg\neg$  is inserted simultaneously in front of every subformula. The various embeddings are logically equivalent. If  $*$  is one of these embeddings, then  $A^* \leftrightarrow A$  classically, and  $A^*$  is provable intuitionistically if and only if  $A$  is provable classically.

Gödel's embedding made it clear that intuitionistic methods went beyond finitism, precisely because abstract notions were allowed. This is clear e.g. from the clause explaining intuitionistic implication in the BHK-interpretation, since there the abstract notion of constructive proof and construction are used as primitives. This fact had not been realized by the Hilbert school until then; Bernays was the first one to grasp the implications of Gödel's result.

Quite important for the proof theory of intuitionistic logic was the formulation in Gentzen's [Gentzen 1935] of the sequent calculi LK and LJ. Using his cut elimination theorem, Gentzen showed that for *IQC* the *disjunction property* DP holds: if  $\vdash A \vee B$ , then  $\vdash A$  or  $\vdash B$ . Exactly the same method yields the *explicit definability* or *existence property* ED: if  $\vdash \exists x A(x)$  then  $\vdash A(t)$

for some term  $t$ . These properties present a striking contrast with classical logic, and have been extensively investigated and established for all the usual intuitionistic formal systems.

The earliest semantics for **IPC**, due to the work of S. Jaskowski, M. H. Stone, A. Tarski, G. Birkhof, T. Ogasawara in the years 1936–1940 was algebraic semantics, with topological semantics as an important special case. In algebraic semantics, the truth values for propositions are elements of a *Heyting algebra* (also known as Brouwerian lattice, pseudo-complemented lattice, pseudo-Boolean algebra, or residuated lattice with bottom). A Heyting algebra is a lattice with top and bottom, and an extra operation  $\rightarrow$  such that  $a \wedge b \leq c$  iff  $a \leq b \rightarrow c$ , for all elements  $a, b, c$  of the lattice. An important special case of a Heyting algebra is the collection of open sets of a topological space  $T$  ordered under inclusion, where  $U \rightarrow V := \text{Interior}(V \cup (T \setminus U))$ . The logical operations  $\wedge, \vee, \rightarrow, \neg$  correspond to the lattice operations  $\wedge, \vee, \rightarrow$  and the defined operation  $\neg a := a \rightarrow 0$ , where 0 is the bottom of the lattice. (A boolean algebra is a special case of a Heyting algebra.)

**5.4. Metamathematics of intuitionistic logic and arithmetic after 1940.** In the early 1940s Kleene devised an interpretation which established a connection between the notion of computable (= recursive) function and intuitionistic logic, namely the *realizability* interpretation.<sup>17</sup>

The essence of the interpretation is that it so to speak *hereditarily* codes information on the explicit realization of disjunctions and existential quantifiers, recursively in numerical parameters. The definition is by induction on the number of logical symbols of sentences (= formulas without free variables): with every formula  $A$  one associates a predicate ‘ $x$  realizes  $A$ ’, where  $x$  is a natural number. Typical clauses are

1.  $n$  realizes  $t = s$  iff  $t = s$  is true.
2.  $n$  realizes  $A \rightarrow B$  iff for all  $m$  realizing  $A$ ,  $n \bullet m$  realizes  $B$ ;
3.  $n$  realizes  $\exists m B(m)$  iff  $n$  is a pair  $(m, k)$ , and  $k$  realizes  $B(m)$ .

Here  $\bullet$  is the operation of application between a number and the code of a partial recursive function. Kleene established the correctness of this interpretation: if  $\mathbf{HA} \vdash A$ , then for some number  $n$ ,  $n$  realizes  $A$ .

The interest of the interpretation is that it makes more true than just what is coded in the formalism **HA**. In particular, the following version of *Church's thesis* may be shown to be realizable:

$$\text{CT}_0 \quad \forall n \exists m A(n, m) \rightarrow \exists k \forall n A(n, k \bullet m)$$

a principle which is easily seen to be incompatible with classical logic. Realizability and its many variants have become a powerful tool in the study of metamathematics of constructive systems.

<sup>17</sup>[Kleene 1945].

We now turn to the further development of truth-value semantics. Algebraic semantics was extended to predicate logic, and A. Mostowski in 1949 was the first one to apply topological models to obtain underivability results for *IQC*. This development culminated in Rasiowa and Sikorski's monograph [Rasiowa and Sikorski 1963]. Although algebraic semantics has proved to be technically useful in metamathematical research, it is so to speak only the algebraic version of the syntax, as witnessed by the fact that *IPC* itself can be made into a Heyting algebra (the *Lindenbaum algebra* of *IPC*). More important from a conceptual point of view are two other semantics, *Beth models*, due to E. W. Beth<sup>18</sup> and *Kripke models*, due to S. Kripke.<sup>19</sup>

Both these semantics are based on partially ordered sets. We call the elements  $k, k', k'' \dots$  of a partially ordered set  $(K, \leq)$  *nodes*. In Kripke models the partial order is arbitrary, in Beth models as defined by Beth it is a finitely branching tree. The interest of these models resides in the intuitive interpretation of the partial order: for Beth models, each node represents a state of information in time, and a higher node represents a possible state of information at a later point in time. The branching of the tree reflects the fact that there are different possibilities for the extension of our knowledge in the future. In the Kripke models, it is more natural to think of the nodes as representing possible stages of knowledge; a higher node in the ordering corresponds to an extension of our knowledge. (That is to say, passing to a later period in time does not force us to move upwards in a Kripke model, only extension of our knowledge does.) In these models one has a notion of ' $A$  is true at  $k$ ', or ' $k$  forces  $A$ '. Falshood  $\perp$  is nowhere forced. It is possible to interpret Beth and Kripke models as topological models for special spaces.

An important aspect of Beth models is the connection with intuitive intuitionistic validity; a formula  $A(R_1, \dots, R_n)$  of *IQC*, containing predicate letters  $R_1, \dots, R_n$  is *intuitionistically valid* if for all domains  $D$  and all relations  $R_i^*$  of the appropriate arity (i.e. the appropriate number of argument places),  $A^D(R_1^*, \dots, R_n^*)$  holds intuitionistically; here  $A^D(R_1^*, \dots, R_n^*)$  is obtained from  $A$  by restricting quantifiers to  $D$ , and replacing  $R_i$  by  $R_i^*$ .

From observations by G. Kreisel in 1958 it follows that, for propositional logic, validity in a Beth model is equivalent to intuitive validity for a collection of propositions  $P_1^\alpha, P_2^\alpha, \dots, P_n^\alpha$  depending on a lawless parameter  $\alpha$  (for the explanation of 'lawless' see the next section).

Beth and Kripke proved completeness for their respective kinds of semantics by classical methods. (Beth originally believed to have also an intuitionistic completeness proof for his semantics.) Veldman was able to show that if one extends the notion of Beth model to *fallible* Beth models, where it is permitted that in certain nodes falshood is forced, it is possible to obtain an intuitionistic

<sup>18</sup>[Beth 1956, 1959].

<sup>19</sup>[Kripke 1965].

completeness proof for Kripke semantics. The idea was transferred to Kripke semantics by H. de Swart. For the fragment of intuitionistic logic without falsehood and negation, fallible models are just ordinary models. For minimal logic, where  $\perp$  is regarded as an arbitrary unprovable proposition letter, one has intuitionistic completeness relative to ordinary Beth models. The best subsequent results in this direction were obtained from work by H. Friedman from ca. 1976.<sup>20</sup>

C. A. Smoryński in [Smoryński 1973] used Kripke models with great virtuosity in the study of the metamathematics of intuitionistic arithmetic.

**5.5. Formulas-as-types.** In essence, the ‘formulas-as-types’ idea (maybe ‘propositions-as-types’ would have been better terminology) consists in the identification of a proposition with the set of its (intuitionistic) proofs. Or stated in another form: in a calculus of typed objects, the types play the role of propositions, and the objects of a type  $A$  correspond to the proofs of the proposition  $A$ .

Thus, since on the BHK-interpretation a proof of an implication  $A \rightarrow B$  is an operation transforming proofs of  $A$  into proofs of  $B$ , the proofs of  $A \rightarrow B$  are a set of functions from (the proofs of)  $A$  to (the proofs of)  $B$ . Similarly, (the set of proofs of)  $A \wedge B$  is the set of pairs of proofs, with first component a proof of  $A$ , and second component a proof of  $B$ . So  $A \wedge B$  corresponds to a cartesian product.

A clear expression of this idea was given in the late 1960s (circa 1968–1969), by W. A. Howard, and by N. G. de Bruijn; H. Läuchli around the same time used the idea for a completeness proof for *IQC* for a kind of realizability semantics.

The analogy goes deeper: one can use terms of a typed lambda calculus to denote natural deduction proofs, and then normalization of proofs corresponds to normalization in the lambda calculus. So pure typed lambda calculus is in a sense the same as *IPC* in natural deduction formulation; similarly, second-order lambda calculus (polymorphic lambda calculus) is intuitionistic logic with propositional quantifiers.

The formulas-as-types idea became a guiding principle in much subsequent research in type theory on the borderline of logic and theoretical computer science. It is used in the type theories developed by P. Martin-Löf to reduce logic to type theory; thus proof by induction and definition by recursion are subsumed under a single rule in these theories. Formulas-as-types plays a key role in the proof-checking language AUTOMATH devised by de Bruijn and collaborators since the late 1960s.

The formulas-as-types idea has a parallel in category theory, where propositions correspond to objects, and arrows to (equivalence classes of) proofs.

<sup>20</sup>See [Troelstra and van Dalen 1988], 2, chapter 13.

J. Lambek investigated this parallel for *IPC* in [Lambek 1972] and later work, culminating in the monograph [Lambek and Scott 1986].

## §6. Intuitionistic analysis and stronger theories.

**6.1. Choice sequences in Brouwer's writings.** As already remarked, the continuum presented a problem to the semi-intuitionists; they were forced to introduce it as a primitive notion, while Brouwer in his thesis tried to explain the continuum and the natural numbers as emanating both from a single 'primeval intuition'.

However, when Brouwer started (circa 1913) with his intuitionistic reconstruction of the theory of the continuum and the theory of point sets, he found that the notion of choice sequence, appearing in Borel's discussion of the axiom of choice (as the opposite, so to speak, of a sequence defined in finitely many words, and therefore in Borel's view of a dubious character) could be regarded as a legitimate intuitionistic notion, and as a means of retaining the advantages of an arithmetic theory of the continuum.

In Brouwer's intuitionistic set theory the dominating concept is that of a spread (in German: 'Menge'). Slightly simplifying Brouwer's original definition, we say that a *spread* consists essentially of a tree of finite sequences of natural numbers, such that every sequence has at least one successor, plus a law  $L$  assigning objects of a previously constructed domain to the nodes of the tree. Choice sequences within a given spread correspond to the infinite branches of the tree. Brouwer calls a sequence  $L(\tilde{\alpha}1), L(\tilde{\alpha}2), L(\tilde{\alpha}3), \dots$ , ( $\alpha$  an infinite branch) an element of the spread. Below we shall use 'spread' only for trees of finite sequences of natural numbers without finite branches, corresponding to the trivial  $L$  satisfying  $L(\tilde{\alpha}(n+1)) = \alpha n$ . Since it is not the definition of a spread, but the way the choice sequences are given to us, which determines the properties of the continuum, we shall henceforth concentrate on the choice sequences.

The notion of spread is supplemented by the notion of species, much closer to the classical concept of set; one may think of a species as a set of elements singled out from a previously constructed totality by a property (as in the separation axiom of classical set theory).

The admissibility of impredicative definitions is not explicitly discussed in Brouwer's writings, though it is unlikely that he would have accepted impredicative definitions without restrictions. On the other hand, his methods allow more than just predicative sets over  $N$ : Brouwer's introduction of ordinals in intuitionistic mathematics is an example of a set introduced by a so-called generalized inductive definition, which cannot be obtained as a set defined predicatively relative to  $N$ .

A *choice sequence*  $\alpha$  of natural numbers may be viewed as an unfinished, ongoing process of choosing values  $\alpha 0, \alpha 1, \alpha 2, \dots$  by the ideal mathematician

(IM); at any stage of his activity the IM has determined only finitely many values plus, possibly, some restrictions on future choices (the restrictions may vary from ‘no restrictions’ to ‘choices henceforth completely determined by a law’). For sequences completely determined by a law or recipe we shall use *lawlike*; other mathematical objects not depending on choice parameters are also called lawlike. An important principle concerning choice sequences is the

*Continuity principle or continuity axiom.* If to every choice sequence  $\alpha$  of a spread a number  $n(\alpha)$  is assigned,  $n(\alpha)$  depends on an initial segment  $\bar{\alpha}m = \alpha 0, \alpha 1, \dots, \alpha(m-1)$  only, that is to say for all choice sequences  $\beta$  starting with the same initial segment  $\bar{\alpha}m$ ,  $n(\beta) = n(\alpha)$ .

This principle is not specially singled out by Brouwer, but used in proofs (for the first time in course notes from 1916-17), more particularly in proofs of what later became known as the bar theorem. From the bar theorem (explained below) Brouwer obtained an important corollary for the finitary spreads, the

*Fan Theorem.* If to every choice sequence  $\alpha$  of a finitely branching spread (fan) a number  $n(\alpha)$  is assigned, there is a number  $m$ , such that for all  $\alpha$ ,  $n(\alpha)$  may be determined from the first  $m$  values of  $\alpha$  (i.e. an initial segment of length  $m$ ).

The fan theorem may be seen as a combination of the compactness of finite trees with the continuity axiom; Brouwer uses the fan theorem in particular to derive:

*The Uniform continuity theorem.* Every function from a bounded closed interval into  $\mathbf{R}$  is uniformly continuous.

The essence of the bar theorem is best expressed in a formulation which appears in a footnote in [Brouwer 1927], and was afterwards used by Kleene as an axiom in his formalization of intuitionistic analysis.

*Bar theorem.* If the ‘universal spread’ (i.e. the tree of all sequences of natural numbers) contains a decidable set  $A$  of nodes such that each choice sequence  $\alpha$  has an initial segment in  $A$ , then for the set of nodes generated by (i)  $X \subset A$ , (ii) if all successors of a node  $n$  are in  $X$ , then  $n \in X$ , it follows that the empty sequence is in  $X$ .

Originally, in [Brouwer 1919], the fan theorem is assumed without proof. In 1923 Brouwer presents an unsatisfactory proof of the uniform continuity theorem, in 1924 ([Brouwer 1924a], Beweis) he proves this theorem via the fan theorem which in its turn is obtained from the bar theorem; his 1924 proof of the bar theorem is repeated in many later publications with slight variations.

Brouwer’s proof of the bar theorem has often been regarded as obscure, but has also been acclaimed as containing an idea of considerable interest. For

in this proof Brouwer analyzes the possible forms of a constructive proof of a statement of the form  $\forall\alpha\exists nA(\bar{\alpha}n)$  for decidable  $A$ .

More precisely, the claim made by Brouwer, in his proof of the bar theorem, amounts to the following. Let  $\text{Sec}(n)$  (' $n$  is secured') mean 'All  $\alpha$  through the node  $n$  pass through  $A$ '. Now Brouwer assumes that a 'fully analyzed proof' of the statement  $\langle \rangle \in \text{Sec}$  or  $\text{Sec}(\langle \rangle)$  (i.e. all  $\alpha$  pass through  $A$ ), can consist of three kinds of steps only:

- (i)  $n \in A$ , hence  $\text{Sec}(n)$ ;
- (ii) if for all  $i$   $\text{Sec}(n * \langle i \rangle)$ , then  $\text{Sec}(n)$ ;
- (iii) if  $\text{Sec}(n)$ , then  $\text{Sec}(n * \langle i \rangle)$ .

Brouwer then shows that steps of the form (iii) may be eliminated, from which one readily obtains the bar theorem in the form given above.

Nowadays the proof is regarded not so much obscure as well as unsatisfactory. As Kleene in [Kleene and Vesley 1965]<sup>21</sup> rightly observes, Brouwer's assumption concerning the form of 'fully analyzed proofs' is not more evident than the bar theorem itself. On the other hand, the notion of 'fully analyzed' proof or 'canonical' proof was used later by M. A. E. Dummett in [Dummett 1977] in his attempts to give a more satisfactory version of the BHK-interpretation of intuitionistic logic.

**6.2. Axiomatization of intuitionistic analysis.** In Heyting's third formalization paper from 1930 we find for the first time a formal statement of the continuity principle. For a long time nothing happened till Kleene in 1950 started working on the axiomatization of intuitionistic analysis; his work culminated in the monograph [Kleene and Vesley 1965]. Kleene based his system on a language with variables for numbers and choice sequences; to arithmetical axioms he added an axiom of countable choice, the axiom of bar induction (equivalent to the bar theorem as formulated above) and a continuity axiom, a strengthening of the continuity principle as stated above ('Brouwer's principle for functions').

The continuity axiom was the only non-classical principle, and Kleene established its consistency relative to the other axioms using a realizability interpretation (function-realizability). He also showed, by means of another realizability notion, that Markov's principle (see 7.2) was not derivable in his system.

In 1963 Kreisel developed an axiomatization based on a language with number variables and two kinds of function variables, for lawlike sequences and for choice sequences. He sketched a proof of the conservativity of the axioms for choice sequences relative to the lawlike part of the system, by means of a translation of an arbitrary sentence into the lawlike part of the

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<sup>21</sup>p. 51.

theory (the ‘elimination translation’). This work finally resulted in [Kreisel and Troelstra 1970].

J. Myhill in 1967 introduced an axiom intended to express the consequences in the language of analysis of Brouwer’s solipsistic theory of the creative subject (as reported above in 4.2), called by him Kripke’s scheme (KS) since Kripke was the first to formulate this principle. KS is

$$\exists \alpha ((\exists x (\alpha x \neq 0) \rightarrow A) \wedge (\forall x (\alpha x = 0) \rightarrow \neg A))$$

for arbitrary  $A$ . KS conflicts with Brouwer’s principle for functions. (Brouwer’s reasoning seems to justify in fact the even stronger  $\exists \alpha (\exists x (\alpha x \neq 0) \leftrightarrow A)$ ). Myhill’s conceptual analysis of the notion of choice sequence is considerably more refined than earlier attempts.

There is an obvious connection between Kripke’s scheme and the theory of the creative subject mentioned in Subsection 4.2. For any proposition  $A$ , the  $\alpha$  in KS may be interpreted as:  $\alpha n \neq 0$  if and only if the creative subject has found evidence for the truth of  $A$  at stage  $n$  of his activity.

Brouwer appears to have vacillated with respect to the precise form which restrictions on choice sequences could take, but in his published writings he does not explicitly consider subdomains of the universe of choice sequences which are characterized by the class of restrictions allowed, except for the trivial example of the lawlike sequences.

In 1958 Kreisel considered ‘absolutely free’ (nowadays *lawless*) sequences ranging over a finitely branching tree, where at any stage in the construction of the sequence no restriction on future choices is allowed; later this was extended to an axiomatization **LS** of the theory of lawless sequences ranging over the universal tree<sup>22</sup> Lawless sequences are of interest because of their conceptual simplicity (when compared to other concepts of choice sequence), as a tool for studying other notions of choice sequence,<sup>23</sup> and because they provide a link between Beth-validity and intuitive intuitionistic validity (cf. 5.4).

As a result of work of Kreisel, Myhill and A. S. Troelstra, mainly over the period 1963–1980, it became clear that many different notions of choice sequence may be distinguished, with different properties.<sup>24</sup>

The publication of [Bishop 1967] led to several proposals for an axiomatic framework for Bishop’s constructive mathematics, in particular a type-free theory of operators and classes,<sup>25</sup> and versions of intuitionistic set theory.<sup>26</sup> Martin-Löf’s type theories have also been considered in this connection.<sup>27</sup>

<sup>22</sup>See [Kreisel 1968].

<sup>23</sup>See e.g. [Troelstra 1983].

<sup>24</sup>For a survey of this topic and its history see [Troelstra 1977], Appendix C; 1983.

<sup>25</sup>[Feferman 1975, 1979].

<sup>26</sup>[Friedman 1977].

As was shown by Aczel in 1978, one can interpret a constructive set theory **CZF** in a suitable version of Martin-Löf's type theory. (**CZF** does not have a powerset axiom; instead there are suitable collection axioms which permit to derive the existence of the set of all functions from  $x$  to  $y$  for any two sets  $x$  and  $y$ ; moreover, the foundation axiom has been replaced by an axiom of  $\in$ -induction). Much of this work is reported in the monograph [Beeson 1985]; Beeson himself made substantial contributions in this area.

**6.3. The model theory of intuitionistic analysis.** Topological models and Beth models turned out to be very fruitful for the metamathematical study of intuitionistic analysis, type theory and set theory. D. S. Scott was the first to give a topological model for intuitionistic analysis, in two papers from 1968 and 1970. In this model the real numbers are represented by the continuous functions over the topological space  $T$  underlying the model. For suitable  $T$ , all real-valued functions are continuous in Scott's model.

This later developed into the so-called sheaf models for intuitionistic analysis, type theory and set theory, with a peak of activity in the period 1977-1984. The inspiration for this development not only came from Scott's models just mentioned, but also from category theory, where W. Lawvere in his [Lawvere 1971] developed the notion of elementary topos — a category with extra structure, in which set theory and type theory based on intuitionistic logic can be interpreted. The notion of elementary topos generalized the notion of Grothendieck topos known from algebraic geometry, and is in a sense 'equivalent' to the notion of an intuitionistic type theory.

Even Kleene's realizability interpretation can be extended to type theory and be recast as an interpretation of type theory in a special topos.<sup>28</sup>

Some of the models studied are mathematically interesting in their own right, and draw attention to possibilities not envisaged in the constructivist tradition (e.g. analysis without an axiom of countable choice, where the reals defined by Dedekind cuts are not isomorphic to the reals defined via fundamental sequences of rationals).

## §7. Constructive recursive mathematics.

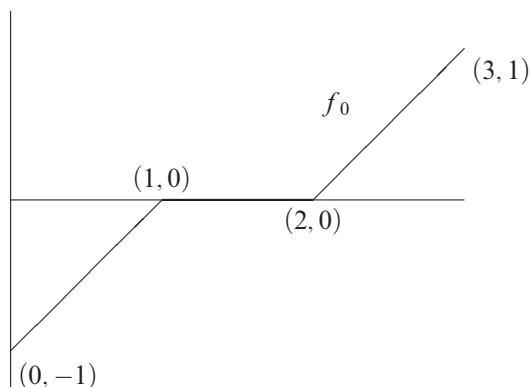
**7.1. Classical recursive mathematics.** Before we can discuss Markov's version of constructive mathematics, it is necessary to say a few words on classical recursive mathematics (RM for short).

In RM, recursive versions of classical notions are investigated, against a background of classical logic. The difference with a more strictly constructive approach is illustrated by the following example:

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<sup>27</sup>Martin-Löf 1975, 1984.

<sup>28</sup>See [Hyland 1982].



The function  $f_a(x)$  given by  $f_a(x) = f_0(x) + a$ , where  $f_0$  is as in the picture, cannot be constructively proved to have a zero as long as we do not know whether  $a \leq 0$  or  $a \geq 0$ . But even classically we can show that  $f_a$  does not have a zero recursively in the parameter  $a$  – and this is typically a result of RM.

Where in constructive recursive mathematics the recursivity in parameters is built into the constructive reading of the logical operators, in RM the recursiveness has to be made explicit.

RM is almost as old as recursion theory itself, since already Turing introduced “computable numbers” in his [Turing 1937], and it is still expanding today. We mention a few examples.

E. Specker was the first to construct an example of a recursive bounded monotone sequence of rationals without a recursive limit.<sup>29</sup> Such sequences are now known as *Specker sequences*.

Kreisel and D. Lacombe constructed singular coverings of the interval in 1957; Kreisel, Lacombe and J. Shoenfield showed, also in 1957, that every effective operation of type 2 is continuous (an effective operation of type 2 is a partial recursive operation with code  $u$  say, acting on codes  $x, y$  of total recursive functions such that  $\forall z (x \bullet z = y \bullet z) \rightarrow u \bullet x = u \bullet y$ ; continuity of  $u$  means that  $u \bullet x$  depends on finitely many values only of the function coded by  $x$ ).

A. I. Maltsev and Y. L. Ershov developed (mainly in the period 1961-1974) the ‘theory of numerations’ as a systematic method to lift the notion of recursiveness from  $\mathbb{N}$  to arbitrary countable structures.<sup>30</sup> In 1974 G. Metakides and A. Nerode gave the first applications of the powerful priority method from recursion theory to problems in algebra.<sup>31</sup> As an example of a striking result we

<sup>29</sup>[Specker 1949].

<sup>30</sup>[Ershov 1972].

<sup>31</sup>See [Metakides and Nerode 1979].

mention the construction of a recursive ordinary differential equation without *recursive solutions*, obtained by M. B. Pour-El and J. I. Richards in 1979.<sup>32</sup>

**7.2. Constructive recursive mathematics.** A. A. Markov formulated in 1948–49 the basic ideas of constructive recursive mathematics (CRM for short). They may be summarized as follows.

1. objects of constructive mathematics are constructive objects, concretely: words in various alphabets.
2. the abstraction of potential existence is admissible but the abstraction of actual infinity is not allowed. Potential realizability means e.g. that we may regard plus as a well-defined operation for all natural numbers, since we know how to complete it for arbitrarily large numbers.
3. a precise notion of algorithm is taken as a basis (Markov chose for this his own notion of ‘Markov-algorithm’).
4. logically compound statements have to be interpreted so as to take the preceding points into account.

Not surprisingly, many results of RM can be bodily lifted to CRM and vice versa. Sometimes parallel results were discovered almost simultaneously and independently in RM and CRM respectively. Thus the theorem by Kreisel, Lacombe and Shoenfield mentioned above is in the setting of CRM a special case of a theorem proved by Tsejtin in 1959: every function from a complete separable metric space into a separable metric space is continuous.

N. A. Shanin<sup>33</sup> formulated a “deciphering algorithm which makes the constructive content of mathematical statements explicit. By this reinterpretation an arbitrary statement in the language of arithmetic is reformulated as a formula  $\exists x_1 \dots x_i A$  where  $A$  is normal, i.e. does not contain  $\forall, \exists$  and the string of existential quantifiers may be interpreted in the usual way. Shanin’s method is essentially equivalent to Kleene’s realizability, but has been formulated in such a way that normal formulas are unchanged by the interpretation (Kleene’s realizability produces a different although intuitionistically equivalent formula when applied to a normal formula).

The deciphering method systematically produces a constructive reading for notions defined in the language of arithmetic; classically equivalent definitions may obtain distinct interpretations by this method. But not all notions considered in CRM are obtained by applying the method to a definition in arithmetic (example: the ‘FR-numbers’ are the reals corresponding to the intuitionistic reals and are given by a code  $\alpha$  for a fundamental sequence of rationals, together with a code  $\beta$  for a modulus of convergence, i.e. relative to a standard enumeration of the rationals  $\langle r_n \rangle_n$  the sequence is  $\langle r_{\alpha n} \rangle_n$ , and  $\forall m m' (|r_{\beta n+m} - r_{\beta n+m'}| < 2^{-k})$ ). An F-sequence is an FR-sequence with the  $\beta$

<sup>32</sup>See [Pour-El and Richards 1989].

<sup>33</sup>[Shanin 1958].

omitted. The notion of an F-sequence does not arise as an application of the deciphering algorithm.)

Markov accepted one principle not accepted in either intuitionism or Bishop's constructivism: if it is impossible that computation by an algorithm does not terminate, then it does terminate. Logically this amounts to what is usually called Markov's principle:

$$\text{MP} \quad \neg\neg\exists x(fx = 0) \rightarrow \exists x(fx = 0) \quad (f : N \rightarrow N \text{ recursive.})$$

The theorem by Tsejtin, mentioned above, needs MP for its proof.

The measure-theoretic paradox (cf. 3.3) is resolved in CRM in a satisfactory way: singular coverings of  $[0, 1]$  do exist, but the sequence of partial sums  $\sum_{n=0}^k |I_n|$  does not converge, but is a Specker sequence; and if the sequence does converge, the limit is  $\geq 1$ , as shown by Tsejtin and I. D. Zaslavskii in 1962.

After about 1985 the number of contributions to CRM considerably decreased. Many researchers in CRM then turned to more computer-science oriented topics.

## §8. Bishop's constructivism.

**8.1. Bishop's constructive mathematics.** In his book 'Foundations of constructive mathematics' the American mathematician E. Bishop launched his programme for constructive mathematics. Bishop's attitude is both ideological and pragmatic: ideological, inasmuch he insists that we should strive for a type of mathematics in which every statement has empirical content, and pragmatic in the actual road he takes towards his goal.

In Bishop's view, Brouwer successfully criticized classical mathematics, but had gone astray in carrying out his programme, by introducing dubious concepts such as choice sequences, and wasting much time over the splitting of classical concepts into many non-equivalent ones, instead of concentrating on the mathematically relevant versions of these concepts. In carrying out his programme, Bishop is guided by three principles:

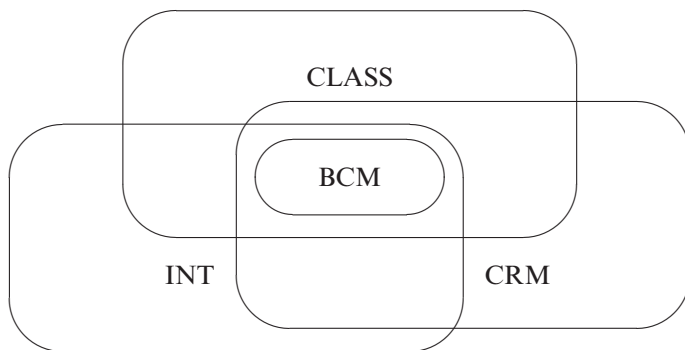
1. avoid concepts defined in a negative way;
2. avoid defining irrelevant concepts — that is to say, among the many possible classically equivalent, but constructively distinct definitions of a concept, choose the one or two which are mathematically fruitful ones, and disregard the others;
3. avoid pseudo-generality, that is to say, do not hesitate to introduce an extra assumption if it facilitates the theory and the examples one is interested in satisfy the assumption.

Statements of Bishop's constructive mathematics (BCM for short) may be read intuitionistically without distortion; sequences are then to be regarded

as given by a law, and accordingly, no continuity axioms nor bar induction are assumed.

Statements of BCM may also be read by a Markov-constructivist without essential distortion; the algorithms are left implicit, and no use is made of a precise definition of algorithm.

Thus BCM appears as a part of classical mathematics, and the situation may be illustrated graphically in the diagram below, where ‘INT’ stands for intuitionistic mathematics, and ‘CLASS’ for classical mathematics.



However, it should not be forgotten that this picture is not to be taken at face value, since the mathematical statements have different interpretations in the various forms of constructivism.

In the actual implementation of his programme, Bishop not only applied the three principles above, but also avoided negative results (only rarely did he present a weak counterexample), and concentrated almost exclusively on positive results.

A steady stream of publications contributing to Bishop's programme since 1967 ensued; among these, two of the most prolific contributors are F. Richman and D. E. Bridges. The topics treated cover large parts of analysis, including the theory of Banach spaces and measure theory, parts of algebra (e.g. abelian groups, Noetherian groups) and topology (e.g. dimension theory, the Jordan curve theorem).

**8.2. The relation of BCM to INT and CRM.** The success of Bishop's programme has left little scope for traditional intuitionistic mathematics; to some extent this is also true of CRM. For all of BCM may at the same time be regarded as a contribution to INT; moreover, in many instances where in INT a routine appeal would be made to a typical intuitionistic result, such as the uniform continuity theorem for functions defined on a closed bounded interval, the corresponding treatment in BCM would simply add an assumption of

uniform continuity for the relevant function, without essential loss in mathematical content. Thus to find scope for specifically intuitionistic reasoning, one has to look for instances where the use of typically intuitionistic axioms such as the continuity axiom or the fan theorem results in a significantly better or more elegant result, and such cases appear to be comparatively rare.

To some extent the above, *mutatis mutandis*, also applies to constructive recursive mathematics. Thus, for example, the usefulness of the beautiful Kreisel-Lacombe-Shoenfield-Tsejtin theorem is limited by two factors: (1) an appeal to the theorem can often be replaced by an assumption of continuity in the statement of the result to be proved; (2) in many cases, continuity without uniform continuity is not enough, as witnessed e.g. by Kushner's example of a continuous function on  $[0, 1]$  which is not integrable.

In the literature contributing to Brouwer's and Markov's programme, a comparatively large place is taken by counterexamples and splitting of classical notions. This may be compared with periods in the development of classical analysis and topology in which there was also considerable attention given to 'pathologies'. In this comparison, BCM exemplifies a later stage in constructivism.

Finally, let us note that in classical mathematics there arise questions of constructivity of a type which has not been considered in the constructivist tradition. For example, one may attempt to find a bound on the *number* of solutions to a number-theoretic problem, without having a bound for the *size* of the solutions.<sup>34</sup>

**§9. Concluding remarks.** In the foundational debate in the early part of this century, constructivism played an important role. Nevertheless, at any time only a handful of mathematicians have been actively contributing to constructive mathematics (in the sense discussed here).

In the course of time, the focus of activity in constructive mathematics has shifted from intuitionism to Markov's constructivism, then to Bishop's constructivism. In addition there has been a steady flow of contributions to classical recursive mathematics, a subject which is still flourishing.

Much more research has been devoted to intuitionistic logic and the meta-mathematics of constructive systems. In this area the work has recently somewhat ebbed, but its concepts and techniques play a significant role elsewhere, e.g. in theoretical computer science and artificial intelligence, and its potential is by no means exhausted (example: the notion of 'formulas-as-types' and Martin-Löf-style type theories).

New areas of application lead to refinement and modification of concepts developed in another context. Thus a development such as Girard's linear

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<sup>34</sup>See [Luckhardt 1989].

logic<sup>35</sup> may be seen as a refinement of intuitionistic logic, obtained by pursuing the idea of a 'bookkeeping of resources' seriously.

A very extensive bibliography of constructivism is [Müller 1987], especially under F50–65.

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# A VERY SHORT HISTORY OF ULTRAFINITISM

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*To the memory of our unforgettable friend Stanley Tennenbaum (1927-2005),  
Mathematician, Educator, Free Spirit.*

In this first of a series of papers on ultrafinitistic themes, we offer a short history and a conceptual pre-history of ultrafinitism. While the ancient Greeks did not have a theory of the ultrafinite, they did have two words, *murios* and *apeiron*, that express an awareness of crucial and often underemphasized features of the ultrafinite, viz. feasibility, and transcendence of limits within a context. We trace the flowering of these insights in the work of Van Dantzig, Parikh, Nelson and others, concluding with a summary of requirements which we think a satisfactory general theory of the ultrafinite should satisfy.

First papers often tend to take on the character of manifestos, road maps, or both, and this one is no exception. It is the revised version of an invited conference talk, and was aimed at a general audience of philosophers, logicians, computer scientists, and mathematicians. It is therefore not meant to be a detailed investigation. Rather, some proposals are advanced, and questions raised, which will be explored in subsequent works of the series.

Our chief hope is that readers will find the overall flavor somewhat “Tennenbaumian”.

**§1. Introduction: The radical Wing of constructivism.** In their *Constructivism in Mathematics*<sup>1</sup>, A. Troelstra and D. Van Dalen dedicate only a small section to Ultrafinitism (UF in the following). This is no accident: as they themselves explain therein, there is no consistent model theory for ultrafinitistic mathematics. It is well-known that there is a plethora of models for intuitionist logic and mathematics: realizability models, Kripke models and their generalizations based on category theory, for example. Thus, a skeptical mathematician who does not feel moved to embrace the intuitionist faith (and most do not), can still understand and enjoy the intuitionist’s viewpoint while remaining all along within the confines of classical mathematics. Model theory creates, as it were, the bridge between quite different worlds.

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<sup>1</sup>[27].

It is hoped that something similar were available for the more radical positions that go under the common banner of Ultrafinitism. To be sure, in the fifteen years since the publication of the above-cited book, some proposals have emerged to fill the void. It is our opinion, though, that nothing comparable to the sturdy structure of model theory for intuitionism is available thus far. This article is the first in a series which aims at proposing several independent but related frameworks for UF.

Before embarking on this task, though, an obvious question has to be addressed first: what is Ultrafinitism, really? As it turns out, a satisfactory answer has proved to be somewhat elusive. A simple answer: all positions in foundations of mathematics that are more radical than traditional constructivism (in its various flavors). But this begs the question. So, what is it that makes a foundational program that radical?

There is at least one common denominator for ultrafinitists, namely the deep-seated *mistrust of the infinite, both actual and potential*. Having said that, it would be tempting to conclude that UF is quite simply the rejection of infinity in favor of the study of finite structures (finite sets, finite categories), a program that has been partly carried out in some quarters<sup>2</sup>.

Luckily (or unluckily, depending on reader's taste), things are not that straightforward, for two substantive reasons:

- First, the rejection of infinitary methods, even the ones based on the so-called potential infinite, must be applied at *all levels*, including that of the meta-mathematics and that of the logical rules. Both syntax and semantics must fit the ultrafinitistic paradigm. Approaches such as Finite Model Theory are simply not radical enough for the task at hand, as they are still grounded in a semantics and syntax that are saturated with infinite concepts<sup>3</sup>.
- Second, barring one term in the dichotomy finite-infinite, is, paradoxically, an admission of guilt: the denier implicitly agrees that the dichotomy itself is valid. But is it? Perhaps what is here black and white should be replaced with various shades of grey.

These two points must be addressed by a convincing model theory of Ultrafinitism. This means that such a model theory, assuming that anything like it can be produced, must be able to generate classical (or intuitionistic) structures, let us call them ultrafinitistic universes, in which an ultrafinitist mathematician can happily live. What this has to mean in practice, if one takes a moment to think about it, is that denizens of those universes should be allowed to treat some finite objects as, *de facto*, infinite. And, indeed, logicians are quite used to the “inside versus outside” pattern of thought; regarding the

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<sup>2</sup>See, for instance, [28].

<sup>3</sup>For example, Trakhtenbrot's Theorem states that first order validity for finite models is not even recursively enumerable. On the other hand the theory of finite *fields* is decidable.

minimal model of  $ZF$ , for example, inside of it countable ordinals look and feel like enormous cardinals. One way of stating our ultimate goal is this: if we could somehow “squeeze” the minimal model below  $\aleph_0$ , we could get what we are looking for.

Only one major obstacle stands in the way: *the apparently absolute character of the natural number series*.

But it is now time for a bit of history . . .

## §2. Short history and prehistory of ultrafinitism.

—*The trouble with eternity is that  
one never knows when it will end.*  
Tom Stoppard, *Rosencrantz and  
Guildenstern Are Dead*.

Ultrafinitism has, ironically, a very long prehistory, encroaching even upon the domains of cultural anthropology and child cognitive psychology. For instance we know that some “primitive” cultures, and also children of a certain age, do not seem to have a notion of arbitrarily large numbers. To them, the natural number series looks a bit like: One, two, three, . . . many! An exploration of these alluring territories would bring us too far afield, so we shall restrict our tale to the traditional beginning of Western culture, the Greeks.

Ancient Greek mathematics does not explicitly treat the ultrafinite. It is therefore all the more interesting to note that early Greek poetry, philosophy, and historical writing incorporate two notions that are quite relevant for the study of the ultrafinite. These notions are epitomized by the two words: *murios* (μυρίος) and *apeirōn* (ἄπειρον).

**2.1. Murios.** The word *murios*, root of the English “myriad”, has two basic senses in ancient Greek writing. These senses are “very many” or “a lot of”; and “ten thousand”. The first sense denotes an aggregate or quantity whose exact number is either not known or not relevant; the second denotes a precise number. With some exceptions, to be given below, the syntax and context make clear which sense is intended in each case. It is part of the aim of this paper to draw attention to the importance of contextualized usage in understanding the ultrafinite.

The earliest occurrences of the term *murios* appear in the oldest extant Greek writing, viz., Homer’s *Iliad* and *Odyssey*. In Homer, all 32 instances of forms of *murios* have the sense “very many” or “a lot of”. Translations often render the word as “numberless”, “countless”, or “without measure”. But what exactly does this mean? Does *murios* refer to an indefinite number or quantity, to an infinite number or quantity, to a number or quantity that is

finite and well-defined but that is not feasibly countable for some reason, or to a number or quantity that the speaker deems large but unnecessary to count? Our investigation reveals that Homer tends to use the term in the last two ways, that is, to refer to numbers or quantities for which a count or measure would be unfeasible, unnecessary, or not to the point. In general, Homer uses the word in situations where it is not important to know the exact number of things in a large group, or the exact quantity of some large mass.

Some representative examples of Homer's usages of *murios*:

- (a) At *Iliad* 2.468<sup>4</sup>, the Achaeans who take up a position on the banks of the Scamander are *murioi* (plural adjective), "such as grow the leaves and flowers in season". The leaves and flowers are certainly not infinite in number, nor are they indefinite in number, but they are not practicably countable, and there is no reason to do so—it is enough to know that there are very many all over.
- (b) In the previous example, the Achaeans must have numbered at least in the thousands. *Murios* can, however, be used to refer to much smaller groups. At *Iliad* 4.434, the clamoring noise made by the Trojans is compared to the noise made by *muriai* ewes who are being milked in the courtyard of a very wealthy man (the ewes are bleating for their lambs). The number of ewes owned by a man of much property would certainly be many more than the number owned by someone of more moderate means, but that rich man's ewes—especially if they all fit in a courtyard—must number at most in the low hundreds. This suggests that the ewes are said to be *muriai* in number because there is a comparatively large number of them; because there is no need to count them (a man who had 120 ewes and was considered very wealthy would not cease to be considered very wealthy if he lost one or even ten of them); and possibly because it might not be practicable to count them (they might be moving around, and they all look rather like one another).

Similarly, at *Odyssey* 17.422 Odysseus says he had *murioi* slaves at his home in Ithaca before he left for the Trojan war. The word for "slaves" in this case is *dmōes*, indicating that these are prisoners of war. Given what we know of archaic Greek social and economic structures, the number of slaves of this type a man in his position could have held must have been in the dozens at most. The key to Odysseus' use of the term is the context. The sentence as a whole reads: "And I had *murioi* slaves indeed, and the many other things through which one lives well and is called wealthy". That is, the quantities of slaves and of other resources that he commanded were large enough to enable

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<sup>4</sup> All translations from the Greek are due to Cherubin. Following the standard form of reference in humanities publishing *Iliad* 2.468 refers to verse 468 of Book Two of the *Iliad*; *Odyssey* 17.422 refers to verse 422 of Book 17 of the *Odyssey*, etc.

him to be considered wealthy. The exact number of slaves might have been countable, but it would have been beside the point to count them.

- (c) *Murios* can also refer to quantities that are not such as to be counted. At *Odyssey* 15.452, a kidnapped son of a king is projected to fetch a *murios* price as a slave. Here *murios* must mean “very large”, “vast”. This is by no means to say that the price will be infinite or indefinite, for a price could not be thus. Rather, the situation is that the exact price cannot yet be estimated, and the characters have no need to estimate it (i.e., they are not trying to raise a specific amount of money).

There are also instances of *murios* in Homer that refer to kinds of things that do not seem to be measurable or calculable. At *Iliad* 18.88, Achilles says that his mother Thetis will suffer *murios* grief (*penthos*) at the death of Achilles, which is imminent. At 20.282, *murios* distress comes over the eyes of Aeneas as he battles Achilles. The usual translation of *murios* here is “measureless”. This translation may be somewhat misleading if it is taken literally, as there is no evidence that the Greeks thought that smaller amounts of grief and distress were necessarily such as to be measurable or measured. A more appropriate translation might be “vast” or “overwhelming”. It is possible that Achilles means that Thetis will suffer grief so vast that she will never exhaust it nor plumb its depths even though she is immortal; but it is also possible that Homer did not consider whether grief or distress could be unending and infinite or indefinite in scope.

The epic poet Hesiod (8th-7th BCE) and the historian Herodotus (5th BCE) sometimes use *murios* in the senses in which Homer does, but they also use it to mean ten thousand. With a very few exceptions, the syntax and context make clear in each instance which meaning is present. At *Works and Days* 252, Hesiod says that Zeus has *tris murioi* immortals (i.e. divinities of various kinds) who keep watch over mortals, marking the crooked and unjust humans for punishment. *Tris* means three times or thrice, and there is no parallel in Greek for understanding *tris murioi* as “three times many” or “three times a lot”. There are parallels for understanding *tris* with an expression of quantity as three times a specific number; and the specific number associated with *murios* is ten thousand. Therefore *tris murios* should indicate thirty thousand.

Some instances of *murios* in Herodotus clearly refer to quantities of ten thousand; some clearly refer to large amounts whose exact quantities are unspecified; and a few are ambiguous but do not suggest any meaning other than these two.

- (d) At 1.192.3, Herodotus says that the satrap Tritantaechmes had so much income from his subjects that he was able to maintain not only warhorses but eight hundred (*oktakosioi*) other breeding stallions and *hexakischiliai kai muriai* mares. *Hexakischiliai* means six times one thousand, so that the whole expression should read six thousand plus *muriai*. The

next line tells us that there are twenty (*eikosi*) mares for every stallion, so that the total number of mares must be sixteen thousand, and *muriai* must mean ten thousand (it is a plural adjective to agree with the noun). The case is similar at 2.142.2-3. Here Herodotus says that three hundred (*triēkosiai*) generations of men come to *muriai* years since three generations come to one hundred (*hekatōn*) years. Clearly, *muriai* means ten thousand here.

- (e) In some places, Herodotus cannot be using *murios* to mean ten thousand, and it is the context that shows this. At 2.37.3, for example, describing the activities of Egyptian priests, he says that they fulfill *muriai* religious rituals, *hōs eipein logōi*. He may in fact mean that they fulfill *muriai* rituals each day, since the rest of the sentence speaks of their daily bathing routines. Herodotus does not give any details about the rituals or their number, and *hōs eipein logōi* means “so to speak”. Thus Herodotus seems to be signalling that he is not giving an exact figure, and *muriai* must simply mean “a great many”. At 2.148.6, Herodotus reports that the upper chambers of the Egyptian Labyrinth *thōma murion pareichonto*, furnished much wonder, so remarkably were they built and decorated. Certainly no particular amount of wonder is being specified here.
- (f) Some occurrences of *murios* are ambiguous in a way that is of interest for the study of the ultrafinite. At 1.126.5, Cyrus sets the Persians the enormous task of clearing an area of eighteen or twenty stadia (2 1/4 or 2 1/2 miles) on each side in one day, and orders a feast for them the next. He tells them that if they obey him, they will have feasts and *muria* other good things without toil or slavery, but that if they do not obey him, they will have *anarithmētoi* toils like that of the previous day. That is, Cyrus is contrasting *muria* good things with *anarithmētoi* bad ones. Is he asking the Persians to consider this a choice between comparable large quantities? If so, a *murios* amount would be *anarithmētos*, which can mean either “unnumbered” or “innumerable”, “numberless”. It is also possible that *murios* is supposed to mean ten thousand, so that the magnitude of the undesirable consequences of defying Cyrus is greater than the great magnitude of the advantages of obeying him. If that is the meaning, Herodotus may be using *murios* in a somewhat figurative sense, as when one says that one has “ten thousand things to do today”.

*Murios*, then, referred in the earliest recorded Greek thought to large numbers or amounts. When it did not refer to an exact figure of ten thousand, it referred to numbers or amounts for which the speaker did not have an exact count or measurement. Our analysis indicates that the speaker might lack such a count or measurement either because the mass or aggregate in question could not

practicably be counted or measured under the circumstances, or because an exact count or measurement would not add anything to the point the speaker was making. In most cases it is clear that the numbers and amounts referred to as *murios* were determinate and finite, and could with appropriate technology be counted or measured. In instances where it is not clear whether that which is referred to as *murios* is supposed to be such as to admit of measuring or counting (Thetis' grief, for example; and Cyrus' *murios* good things if they are comparable to the *anarithmētos*), there is no evidence as to whether the *murios* thing or things are supposed to be infinite or indefinite in scope. Indeed, there is no evidence that these early writers thought about this point. (This is perhaps why *anarithmētos* can mean both "unnumbered" and "innumerable", and why it is often difficult to tell which might be meant and whether a writer has in mind any distinction between them.)

When *murios* does not mean "ten thousand", context determines the order of quantity to which it refers. Any number or amount that is considered to be "a lot" or "many" with respect to the circumstances in which it is found can be called *murios*. Leaves and flowers in summer near the Scamander number many more than those of other seasons, perhaps in the millions; but the rich man's ewes are *muriai* too, even if they number perhaps a hundred. They are several times more than the average farmer has, and they may fill the courtyard so much that they cannot easily be counted.

In this way groups and extents that would be acknowledged to be finite and perhaps effectively measurable or countable under some circumstances would be called *murios* when actual circumstances or purposes made counting or measuring impossible, impractical, or unnecessary. This step would be equivalent to treating finite things as *de facto* infinite. This freedom, we hold, is precisely what a convincing model theory of Ultrafinitism should allow us to do.

**2.2. Apeirōn.** Since *murios* seems to refer overwhelmingly to determinate and finite quantities, it is useful to note that Greek had ways of referring to quantities that were indeterminate, unlimited, indefinite, or infinite. The most significant of these, for our purposes, was the word *apeiros* or *apeirōn* (m., f.)/*apeiron* (n.).

The etymology of this word is generally understood to be *peirar* or *peras*, "limit" or "boundary", plus alpha privative, signifying negation: literally, "not limited" or "lacking boundary"<sup>5</sup>. Etymology alone does not tell us the range of uses of the term or the ways in which it was understood, so we must again consider its occurrences in the earliest sources. The term appears as early as Homer, in whose poems it generally refers to things that are vast in extent, depth, or intensity.

<sup>5</sup>For an excellent review, see [21, pages 67-70]. See also [14, pages 231-239].

Homer uses *apeiros* most frequently of expanses of land or sea. In each case, the *apeiros/apeirōn* thing is vast in breadth or depth; whether its limits are determinable is not clear from the context, but limits do seem to be implied in these cases. Some instances may imply a surpassing of some sort of boundaries or borders (though not necessarily of all boundaries or borders). At *Iliad* 24.342 and *Odyssey* 1.98 and 5.46, a god swiftly crosses the *apeirōn* earth. Within the context, it is clear that the poet means that the divinity covers a vast distance quickly. There may be a further implication that the gods transcend or traverse boundaries (be these natural features or human institutions) with ease, so that the world has no internal borders for them. Similarly, in *Odyssey* 17.418, the expression *kat' apeirona gaian*, often translated as “through[out] the boundless earth”, is used to suggest that something is spread over the whole earth. What is spread covers a vast expanse, and it also crosses all boundaries on the earth.

Two other Homeric examples are of interest. At *Odyssey* 7.286, a sleep is described as *apeirōn*, meaning either that it is very deep, or unbroken, or both. At *Odyssey* 8.340, strong bonds are *apeirōn*, surpassing limits of a god's strength, and so unbreakable.

Hesiod also uses *apeirōn* to describe things that extend all over the earth, but also uses the word once in reference to a number. In *Shield of Heracles* 472, the word refers to a large number of people from a great city involved in the funeral of a leader; the sense seems to be that there were uncountably many, and possibly that the leader's dominion had been vast.

Herodotus (5th century BCE) uses *apeirōn* in two cases where its meaning clearly derives from the privative of *peirar*<sup>6</sup>. In 5.9 he uses it to refer to a wilderness beyond Thracian settlements. In 1.204 a plain is *apeiron*, perhaps hugely or indeterminately vast. In both cases, Herodotus knows that the lands are finite in extent (he identifies the peoples who live beyond them). The contexts suggest that he means that these lands are vast and that their exact boundaries are not known. He may also have in mind that they cannot be easily, if at all, traversed by humans.

The first and perhaps best-known philosophical use of *apeiron* is in the reports about the work of Anaximander's in the sixth century BCE. Anaximander is reported to have held that the source of all familiar things, the fundamental generative stuff of the cosmos, was something *apeiron*. The testimonia report that the *apeiron* was eternal in duration, unlimited or indeterminate in extent, and qualitatively indeterminate.

All of the familiar cosmos, for Anaximander, arose from the *apeiron*.

In his [14], Kahn holds that Anaximander's *apeiron* is “primarily a huge, inexhaustible mass, stretching away endlessly in every direction”. The *apeiron*

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<sup>6</sup>Herodotus also uses a homophone word that is derived from another root, so we have only included instances where context clearly indicates that the word is the one derived from *peirar*.

surely must be at least that, but there is no reason to think that it is primarily that. As McKirahan [19] notes, the discussions of Anaximander in Aristotle and the Peripatetics make clear that the *apeiron* must also be a stuff of indefinite kind or quality. It must be this because it is supposed to be able to give rise to every kind of thing, and because (according to Aristotle in *Physics* Gamma 3) (see [3]) if any one kind of thing, e.g. fire, was *apeiron* it would overcome and destroy everything else. Clearly that has not happened. The question would then seem to arise as to why the indefinite stuff of unlimited extent does not overwhelm all specific stuffs, and result in a universe that is wholly indefinite, and so not a cosmos, i.e. an ordered universe. The answer to this question may perhaps be found in Anaximander's contention that the *apeiron* is fundamentally unstable. It is indeterminate even in its state. According to the ancient reports, the *apeiron* was supposed to be always in motion. Through this, somehow, "opposites" (hot and cold, wet and dry, light and dark, perhaps others) separate off and interact to form the world of familiar things. Eventually, according to the only apparent quotation we have from Anaximander, the things or opposites "pay penalty and restitution to one another for their injustice, according to the arrangement of time", and perish back into the *apeiron*, whence the cycle begins anew. (What the "injustice" is remains a subject of much speculation; the word used suggests that it may be some sort of imbalance or encroachment.) It is worth noting that for Anaximander the whole cosmos may be at the same stage in the cycle at any given time, but that is not the only possibility. It is also possible that different parts of the cosmos are at different stages in the cycle, so that qualitative and quantitative indeterminacy are present in some regions and not in others, and the whole therefore remains *apeiron* in some respects.

We may note that so far no instance of *apeirōn* clearly meant "infinite". Only one, Anaximander's, could possibly involve an infinite extent, and even in that case it is not clear that the extent is infinite; it may be indefinite or inexhaustible without being infinite. Anaximander's stuff is eternal, i.e. always in existence, but it is not at all clear that a sixth century Greek would have taken "always" to mean an infinite amount of time. Whether any Greek of the 8th to 5th centuries BCE conceived of quantities or magnitudes in a way that denoted what we would call infinity is not certain.

It is sometimes thought that Zeno of Elea (5th century BCE) spoke of the infinite, but there is good evidence that he had quite a different focus. It is only in the arguments concerning plurality that are preserved by Simplicius that we find what may be quotations from Zeno's work (regarding his arguments concerning motion and place we have only reports and paraphrases or interpretations)<sup>7</sup>. In fragments DK29 B1 and B2, Zeno argued "from saying

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<sup>7</sup>It is possible that some of Zeno's paradoxes of motion dealt with infinitely long sequences of steps. Aristotle suggests that they did. Aristotle used the word *apeiron* to describe these

that multiple (*polla*, many) things are, saying opposite things follows". In particular, if we say that multiple things are, then we must conclude that "the same things must be so large as to be *apeira* (neuter plural) and so small as to lack magnitude (*megethos*)". Zeno was evidently interested in the claim that there are multiple things with spatial magnitude, and it appears from the fragments that he thought that the possibilities for analyzing the components of spatial magnitude were that a thing that has spatial magnitude must be composed of parts with positive spatial magnitude, parts of no magnitude, or some combination of these. If a thing had no magnitude, Zeno argued, it would not increase (in magnitude) anything to which it was added, nor decrease anything from which it was removed. Therefore it could not "be" at all (at least, it could not "be" as the spatial thing it was said to be). Nothing with magnitude could be composed entirely of such things. However, if we assume that the components of a spatial thing have positive magnitude, another problem arises. In measuring such a thing, we would try to ascertain the end of its projecting part, (i.e. the outermost part of the thing). Each such projecting part would always have its own projecting part, so that the thing would have no ultimate "extreme" (*eschaton*). That is, the outer edge of something always has some thickness, as do the lines on any ruler we might use to measure it; and this thickness itself can always be divided. Thus the magnitude of a spatial thing, and thus its exact limits, will not be determinable. There is nothing in this to suggest that Zeno thought that the claim that there are multiple spatial things led to the conclusion that such things must be infinitely large. Rather, his description suggests that the things would be indeterminable, and indeterminate or indefinite, in size. They would also be *apeira*, indeterminate or indefinite, in number.

What of Zeno's paradoxes of motion? Modern interpretations of them generally present them as dealing with infinite sequences of steps, or with distances or times that seem to be infinite. Unfortunately, our evidence concerning the paradoxes of motion does not include any quotations from Zeno, but only reports that are at best second-hand. We cannot tell whether Zeno used the term *apeiron* in any of them, much less whether he used the term to refer to the infinite. Still, Aristotle's discussion of these paradoxes in the *Physics*, our earliest report, is replete with information that is germane to our purposes here.

In *Physics* Book Zeta Chapter 2, Aristotle says that "Zeno's argument is false in taking the position that it is not possible to traverse *apeira* things or to touch each of (a collection of) *apeira* things in a *peperasmenos* time"<sup>8</sup>. *Apeirōn*

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sequences, but it is not known whether Zeno did. See Aristotle, *Physics* Z2 [3]. See [26] for Simplicius's quotations from Zeno; some translations appear in [19].

<sup>8</sup>The grammar suggests that Aristotle means that according to Zeno it is not possible to traverse any group of things that is *apeiron*, not necessarily a group of things each of which is *apeiron*.

here is usually translated as “infinite”, but Aristotle sometimes uses the term to mean “unlimited” or “indefinite” or “indeterminate”, and he may intend those senses here. Similarly, *peperasmēnon* is often translated as “finite”, but it may equally well mean “limited”, “definite”, or “determinate”. Aristotle’s response to the position he attributes to Zeno is to note that “both [linear] length and time, and anything continuous, are said (spoken of) in two ways: with respect to division and with respect to extremities. Therefore while it is not possible in a *peperasmēnos* time to touch things that are *apeiron* with respect to quantity, it is possible to do so if they are *apeiron* with respect to division, for time (or: a time, an interval of time) is itself *apeiros* in this way . . . things that are *apeiron* are touched not by a *peperasmēnos* time but by *apeiros* time” (233a22-35).

It will immediately be seen that Aristotle’s argument works equally well whether *apeiron* and *peperasmēnon* mean respectively “unlimited” and “limited”, “infinite” and “finite”, or “indeterminate” and “determinate”. In Aristotle’s attempt to find a coherent account of the motions and changes of distinct things, infinity, indeterminacy, and unlimitedness each pose challenges. The one perhaps most familiar to us is that of the possibility that it would take an infinite number of steps or stages, and so conceivably an infinite stretch of time, to traverse a distance of finite length. Other problems would arise for the prospects of coherence and explanation if it turned out that an indefinite number of steps would be needed to cross a definite distance, or if a determinate distance were to turn out to be composed of an unlimited number of smaller intervals (of possibly indeterminate length). For if it would take an indefinite or unlimited number of steps, and hence an indefinite or unlimited number of time intervals, to cross a defined distance, how could we tell when, and therefore if, we had completed the crossing? And if the presumed definite distance turned out to be composed of an indefinite number of intervals of positive magnitude, how could we determine where, and hence if, it began and ended? Moreover, if we could not identify exactly where objects and intervals of spatial magnitude began or ended, could we say with consistency that there are the multiple, distinct objects that are necessary (for the Greeks at least) to an account of motion?

When Aristotle returns to Zeno in Chapter 9 of Book Zeta, he describes in brief terms the four paradoxes known to us as the Dichotomy, the Achilles, the Arrow, and the Stadium or Moving Rows. Aristotle indicates that he has already discussed the Dichotomy, and his description of it matches the challenge of Zeno’s that he had discussed in Chapter 2. In his descriptions of the remaining three paradoxes, the word *peperasmēnon* appears only in the Achilles, and *apeiron* does not appear at all, though the concept is implicit in the discussion. The problem of the Achilles has to do with how a slower runner with a head start will be overtaken by a swifter runner, if the swifter runner

must first reach each point that the slower one had reached (239b15-30). Aristotle argues that as long as the pursued runner must traverse a limited or finite (*peperasmēnēn*) distance, the faster pursuer can overtake him. The defect with Zeno's alleged claim that the pursuer will not overtake the pursued, Aristotle says, is the way in which Zeno divides the magnitude (distance) between the runners. Aristotle seems to have in mind that Zeno is taking the distance between the runners as always further divisible rather than as composed of pieces of definite size that can be matched by steps of definite size. Here too, *peperasmēnēn* could refer either to finitude or to limitedness, determinacy, of the distance.

Let us not lose sight, however, of Aristotle's careful locution from Chapter 2: continuous things are spoken of in two ways. With respect to division, continuous things of definite length are nonetheless *apeira*. Aristotle thus speaks of things as *apeiron* or *peperasmēnon* within a certain context or in a certain respect or for a certain purpose. Where Aristotle may well have been at odds with Zeno, as we see from the remark about how Zeno looked at distance in the Achilles, was precisely over context or purpose. That is, they seem to have had different projects in mind: for Aristotle, providing an account of the things we say move and change; for Zeno, understanding whether we could have a coherent account of what is if we say that that includes discrete things of positive magnitude. It may be as well that the two philosophers differed over the question of the contexts we need to invoke in order to understand what is. One of the most important points, then, to take from the discussions of Zeno in Aristotle's *Physics* is Aristotle's care in distinguishing the aspects of a thing that are *apeiron* and those that are *peperasmēnon* under each set of conditions or in each context.

The view we have presented of Zeno's concerns finds additional support in his extant fragments. In the fragments on multiplicity mentioned earlier (DK29 B1 and B2), we have seen that Zeno argued that if multiple spatial things are, they must be both so large as to be *apeira* and so small as to have no magnitude. We have already discussed why a spatial (as opposed to geometrical) object composed of parts that have no magnitude would pose problems. We have seen why Zeno might find difficulties with the prospect of spatial things that each had *apeiron* magnitude: one would not be able to establish or support the claim that there were distinct spatial things at all. It remains to be seen, then, why Zeno would say specifically that if we say that there are things with spatial magnitude, those things will turn out to have both no magnitude and *apeiron* magnitude.

Recall that on Zeno's analysis, it appeared that only if a thing had no magnitude could it have limited or finite magnitude. The only way for it to have positive magnitude, it seemed, was for it to have parts of *apeiron* magnitude. Someone might then respond that perhaps there was a way for

things to have limited positive magnitude: perhaps the inner regions of such a thing would have positive and thus *apeiron* magnitude, and the surfaces or ends or edges would have no magnitude, and serve as the limits. Zeno would not accept such a solution. First, he would say, the outer parts would add no magnitude, so that the wholes of things would still have *apeiron* magnitude. Second, the two kinds of components of things would be impossible as definite spatial objects and as parts thereof.

There are two more fragments that make Zeno's concerns clearer, and that we can now see show somewhat more of an emphasis on limit, determinacy, and their opposites than on what we would term finitude and infinity. DK29 B3 supports the hypothesis that Zeno was concerned about the coherence of an account of what is that invoked distinctness. In this fragment Zeno argued that if many things are, they must be both *peperasmēnon* in number, for they are as many as they are; and also *apeiron*, because something must be between any two, else those two would not be separate. That would imply that we cannot tell how many things are present in any area at any time, nor can we tell where (or thus if) any of them begins or ends.

We have no evidence that Zeno concluded from this that only one thing is. Simplicius, our main source for his fragments, claims that Zeno concluded that only one thing is, but does not furnish any quotations in which Zeno says such a thing. Moreover, in DK29 A16, Eudemus reports that Zeno said that if anyone could show him what the one is, he will be able to tell the things that are. In other words, Zeno did not think that to say that one thing is would be any more coherent or understandable than to say that many things are<sup>9</sup>.

A more extensive discussion of these matters in Zeno is beyond the scope of the present paper, and is available in [10] and [9].

In the philosophy of the fourth century BCE, and arguably as early as Zeno, an *apeirōn* quantity could not be calculated exactly, at least as long as it was regarded from the perspective according to which it was *apeirōn*. In fact, Aristotle's argument that a continuous magnitude bounded at both ends could be traversed in a finite amount of time—despite the fact that it contains, so to speak, an *apeirōn* number of points, and despite Zeno's Dichotomy argument—rests precisely on the notions that the magnitude is not composed of the *apeirōn* number of points, and that from one perspective it is bounded. Aristotle does not refute Zeno's argument, but merely argues that within the framework of his physics, the question Zeno addresses can be put differently. Thus where *myrios* did not clearly refer to ten thousand, a *myrios* quantity was generally recognized as definite but was not calculated exactly. An *apeirōn* thing or quantity in Homer or Herodotus might be definite or not, and in later thinkers, especially in philosophy, the term came to emphasize that aspect of the thing or group or quantity that was indefinite, indeterminate, or

<sup>9</sup>A more complete account can be found in [9].

unlimited. Thus we find the indeterminacy, indefiniteness, or unlimitedness applying with respect to some context or conditions (practical or conceptual).

This concludes our remarks on ultrafinitistic themes in Greek thought. Our subsequent reflections on the ultrafinite will orbit, for the most part, around the *murios-apeirōn* pair, as if around a double star.

We now skip over two thousand years of mathematical and philosophical thought—where ultrafinitistic themes do crop up from time to time—picking up the thread once again well into the twentieth century.

**§3. Recent history of UF.** The passage from the prehistory to the history of UF is difficult to trace. Perhaps a bit arbitrarily, we shall say that it begins with the criticism of Brouwer's Intuitionist Programme by Van Dantzig in 1950<sup>10</sup>.

According to this view, an infinite number is a number that surpasses any number a person can cite. One is here reminded of a game inadvertently invented by the Greek mathematician Archimedes in his *Sand Reckoner*<sup>11</sup>. A game that is still played to this day, it is as follows: two players, A and B, try to outdo each other at naming large numbers. The contestant who is able to construct what is essentially, in contemporary jargon, a faster growing primitive recursive function, wins the game. The winner is the (temporary) owner of the so-called infinite numbers<sup>12</sup>.

A second major character in this story is the Russian logician Yessenin-Volpin. In a series of papers<sup>13</sup> he presents his views on UF. Unfortunately, in spite of their appeal, his presentation can be at times obscure<sup>14</sup>. In any case, one of the fundamental ideas put forth by Volpin is that there is no uniquely defined natural number series. Volpin's attack attempts to unmask the circularity behind the induction scheme, and leaves us with various non-isomorphic finite natural number series. He also argues for the idea that there is a sense in which even small finite numbers can be considered infinite.

A few years later, one morning in fall the autumn of 1976 to be precise, the Princeton mathematician Ed Nelson had what might be described as an ultrafinitistic epiphany<sup>15</sup>, losing his "pythagorean faith" in the natural numbers. What was left was nothing more than finite arithmetic terms, and the rules to manipulate them. Nelson's *Predicative Arithmetic* (see [23]) was the result.

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<sup>10</sup>See [11]. Van Dantzig himself points out that some of his ideas were anticipated by the Dutch philosopher Mannoury, and by the French mathematician Emil Borel (see [7]). For example, Borel observed that large finite numbers (*les nombres inaccessibles*) present the same difficulties as the infinite.

<sup>11</sup>For a good translation see [2].

<sup>12</sup>See on this also [4].

<sup>13</sup>See for instance his 1970 manifesto [32].

<sup>14</sup>Though David Isles has made a serious and quite successful attempt to clarify some of Volpin's tenets in [13].

<sup>15</sup>See his [22].

That text, the product of his epiphany and an essential step toward the re-thinking of mathematics along strictly finitist lines, seems to us however to fall short of Nelson's amazing vision. For example, why stop at induction over bounded formulae? If the infinite number series is no more, and arithmetic is just a concrete manipulation of symbols (a position that could be aptly called ultra-formalist), "models" of arithmetics are conceivable, where even the successor operation is not total, and all induction is either restricted or banished altogether.

The next milestone we take note of is Rohit Parikh's 1971 "Existence and Feasibility in Arithmetics", [24]. This paper introduces a version of Peano Arithmetic enriched with a unary predicate  $F$ , where the intended meaning of the statement  $F(x)$  is that  $x$  is *feasible*. Mathematical induction does not apply to formulas containing the new predicate symbol  $F$ . Moreover, a new axiom is added to Peano Arithmetic expressing that a very large number is not feasible. More precisely, the axiom says that the number  $2_{1000}$ , where  $2_0 = 1$  and  $2_{k+1} = 2^{2_k}$ , is not feasible. Parikh proves that the theory  $PA + \neg F(2_{1000})$  is *feasibly consistent*: though inconsistent from the classical standpoint, all proofs of the inconsistency of this theory are unfeasible, in the sense that the length of any such proof is a number  $n \geq 2_{1000}$ .

From the point of view of these reflections, Parikh achieves at least two goals: first, he transforms some ultrafinitistic claims into concrete theorems. And secondly, he indicates the way toward an ultrafinitistic proof theory.

Parikh's approach has been improved upon by several authors. Quite recently, Vladimir Sazonov in his [25] has made a serious contribution toward making explicit the structure of Ultrafinitistic Proof Theory. In the cited paper the absolute character of being a feasible number is asserted, on physicalistic grounds<sup>16</sup>. For our part, though, physicalistic explanations are less than convincing. As we have pointed out elsewhere in this paper, we believe that maintaining the notion of contextual feasibility is important. After all, who really knows what is the nature of the universe? Perhaps new advances in physics will show that the estimated upper bound of particles in the universe was too small. But whereas logic should be able to account for physical limitations, it should not be enslaved by them<sup>17</sup>.

Parikh's 1971 paper, groundbreaking as it was, still leaves us with a desire for more: knowing that  $PA + \neg F(2_{1000})$  is feasibly consistent, there ought to be *some* way of saying that it has a model. In other words, the suspicion arises that, were a genuine semantics for ultrafinitistic theories available, then Gödel's completeness theorem (or a finitist version thereof) should hold true

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<sup>16</sup>Sazonov's articulated position on this issue is more subtle, as can be seen from his recent FOM postings.

<sup>17</sup>For further work on Parikh's approach see also [8], in which Alessandra Carbone and Stephen Semmes have investigated the consistency of  $PA + \neg F(2_{1000})$  and similar theories from a novel proof theoretical standpoint, involving the combinatorial complexity of proofs.

in some form. But where to look for such a semantics? Models are structured sets, or, alternatively, objects in some category with structure. We must thus turn from proof theory to set theory and category theory.

On the set-theoretical side, there are at least two major contributions. The first one is Vopenka's proposal to reform, so to speak, Cantorian set theory, known as Alternative Set Theory, or AST (see [31]). AST has been developed for more than three decades, so even a brief exposition of it is not possible here. In broad outline, AST is a phenomenological theory of finite sets. Some sets can have subclasses that are not themselves sets, and such sets are infinite in Vopenka's sense. This calls to mind one of the senses of the word *apeiron*, as previously described: some sets are (or appear) infinite because they live outside of our perceptual horizon. It should be pointed out that AST is not, per se, a UF framework. However, Vopenka envisioned the possibility of "witnessed universes", i.e. universes where infinite (in his sense) semisets contained in finite sets do exist. Such witnessed universes would turn AST into a universe of discourse for ultrafinitism. To our knowledge, though, witnessed AST has not been developed beyond its initial stage.

Other variants of set theory with some finitist flavor have been suggested. Andreev and Gordon in their [1], for example, describe a theory of Hyperfinite Sets (THS) which, unlike Vopenka's, is not incompatible with classical set theory. Interestingly, both AST and THS produce as a by-product a natural model of non-standard analysis<sup>18</sup>, a result which should be of interest to mainstream mathematicians.

The second set-theoretical approach of which we are aware, is that described in Shaughan Lavine's [15]. Here, a finitistic variant of Zermelo-Frankel set theory is introduced, where the existence of a large number, the Zillion, is posited. The reader may recall an idea which we hope is, by now, a familiar one, namely that of *murios*. Here the number Zillion replaces the missing  $\aleph_0$ .

We move finally, to category theory. From our point of view this is, with one notable exception, an uncharted, but very promising, area. The single exception is the work of the late Jon M. Beck, involving the use of simplicial and homotopic methods to model finite, concrete analysis (see for instance [5, 6]). As we understand it, Beck's core idea is to use the simplicial category  $\Delta$ , truncated at a certain level  $\Delta[n]$ , to replace the role of the natural number series—or, because we are here in a categorical framework, the so-called natural number object that several topoi possess. The truncated simplicial category has enough structure to serve as a framework for some finitistic version of recursion; moreover, its homotopy theory provides new tools to model finite flow diagrams. As has been pointed out by Michael Barr, addition for the

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<sup>18</sup>The very large and the very small are indeed intimately related: if one has a consistent notion of large, unfeasible number  $n$ , one automatically gets the infinitesimal  $\frac{1}{n}$ , via the usual construction of the field of fractions  $\mathcal{Q}$ .

finite calculator is not an associative operation. But homotopy repairs the lack of associativity by providing associativity up to homotopy via coherence rules.

As is well known, topoi have an internal logic which is intuitionistic. One hopes that by isolating feasible objects in the realizability topos via a suitable notion of *feasible realizability*, a categorical universe of discourse for UF could be, as it were, carved out.

In conclusion:

- First, the notion of feasibility should be *contextual*. An object such as a term, a number, or a set is feasible only within a specific context, namely one which specifies the type of resources available (functions, memory, time, etc). Thus a full-blown model theory of UF should provide the framework for a dynamic notion of feasibility.
- As the context changes, so does the notion of feasibility. What was unfeasible before, may become feasible now. Perhaps our notion of potential infinity came as the realization, or belief, that *any* contest can be transcended.
- Degrees of feasibility, so to speak, are not necessarily linearly ordered. One can imagine contexts in which what is feasible for A is not feasible for B, and vice versa.
- Last, but not least, as to the *murios-apeiron* pair: Every convincing approach to UF should be broad enough to encompass both terms. We saw above that any number or amount termed “many” with respect to the circumstances in which it is found, is *murios*. Pseudo-finite model theory, namely the restriction of first-order logic to models with the property that every first-order sentence true in the model is true in a finite model<sup>19</sup>, captures one aspect of this idea. With pseudo-finite models particular properties or artifacts of the structure, such as its cardinality, are “divided out”, so to speak. This means then that the pseudo-finite structure instantiates a general notion of finiteness, somewhat similar to uses of the word *murios* in Homer as cited above: the unspecified or indefinite (but finite) “many”. The notion of pseudo-finite structure is far from what is aimed at here, as pseudo-finite structures are structures which are elementarily equivalent to ultraproducts of finite structures. They are, in a sense, too “sharp” for what is at stake here. However, they do point in the right direction, in that they show that even in classical logic the very notion of finiteness is to some extent blurred, if one suitably restricts the underlying logic. This hint, as we shall show in a future work, is indeed a pivotal one.

We hope that our reflections on the *murios/apeiron* pair go some way toward shedding some light on these issues. Our belief is that every convincing

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<sup>19</sup>See for example [29].

approach to Ultrafinitism should include the notions of contextual uncountability, of indefiniteness, and of traversing limits. Even better, any such approach should unify these two streams of thought into a single, flexible framework.

This concludes our Very Short History. In our next paper, we shall offer a proposal which strives to take all the mentioned points into account.

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## SUE TOLEDO'S NOTES OF HER CONVERSATIONS WITH GÖDEL IN 1972-5<sup>†</sup>

SUE TOLEDO

### Husserl.

**Conversation March 3, 1972.** Husserl's philosophy is very different before 1909 from what it is after 1909. At this point he made a fundamental philosophical discovery, which changed his whole philosophical outlook and is even reflected in his style of writing. He describes this as a time of crisis in his life, both intellectual and personal. *Both* were resolved by his discovery. At this time he was working on phenomenological investigation of time.

There is a certain moment in the life of any real philosopher where he for the first time grasps directly the system of primitive terms and their relationships. This is what had happened to Husserl. Descartes, Schelling, Plato discuss it. Leibniz described it (the understanding or the system?) as being like the big dipper — it leads the ships. It was called understanding the absolute.

The analytic philosophers try to make concepts clear by defining them in terms of primitive terms. But they don't attempt to make the primitive terms clear. Moreover, they take the wrong primitive terms, such as "red", etc., while the correct primitive terms would be "object", "relation", "well", "good", etc.

The understanding of the system of primitive terms and their relationships cannot be transferred from one person to another. The purpose of reading Husserl should be to use his experience to get to this understanding more quickly. ("Philosophy As Rigorous Science" is the first paper Husserl wrote after his discovery.)

Perhaps the best way would be to repeat his investigation of time. At one point there existed a 500-page manuscript on the investigation (mentioned in letters to Ingarden, with whom he wished to publish the manuscript). This manuscript has apparently been lost, perhaps when Husserl's works were taken to Louvain in 1940. It is possible that this and other works were removed.<sup>1</sup>

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<sup>†</sup>Typed and edited by Juliette Kennedy, from a xerox copy of Toledo's hand-written notes. No additions or deletions were made to these notes except for the addition of punctuation, as indicated.

<sup>1</sup>We take the liberty of making one exception to our policy of not annotating these notes, as follows: As to Husserl's lost manuscript on time, Gödel made similar remarks, at various times, about manuscripts disappearing (e.g. some of Leibniz's). And though such remarks were usually

In particular there are almost no examples of fully worked out phenomenological investigations from after 1909, the works of Husserl that have been published being basically a discussion of what to do, a little about how to do it, etc. Even the book on time is almost completely from before 1909, and perhaps even taken from Heidegger's lecture notes.

The concept of "world", (the idea of "object", "existence"), or rather of "objective existence" are the central ones.

There are probably different ways to find these primitive terms. The method of phenomenology is not to investigate the terms themselves, but rather to investigate how we handle them. (Husserl never mentions that his goal for phenomenology is finally to come to an understanding of the primitive terms themselves.)

Another thing he doesn't mention is that when we investigate our way of handling these concepts, there is no reason to assume that we always handle them correctly. If one considers the early age at which we start, it would not be surprising that would make many wrong combinations of the primitive terms. (That is why critical philosophy as called critical philosophy—note that Husserl was a follower of Kant in his second period but not in his first.)

As an example of criticism of our concepts, Kant says we have a wrong idea of time, that it is really a form of our intuition, while we conceive it as being in the world, in things. (Actually there is a kind of double talk in Kant. He says that space and time don't exist objectively, but that our thoughts about them are empirically true. He means that we have a wrong idea about time & space—actually partially correct and partially incorrect, but that we apply

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greeted with skepticism, in this case at least, Gödel was completely correct. The following is a statement of Husserl's former student Roman Ingarden from 1962:

Thus in 1927 Husserl proposes to me also that I should adjust a great bundle of manuscripts (consisting probably of 600–700 sheets of paper) on the original constitution of time, which he had written in Bernau in 1917–1918. He gave me a completely free hand with the editing of the text, his only condition being that the work should be published under our two names. I could not, however, accept his proposition, first of all because I was convinced that Husserl would have done the work much better himself at the time. To tell the truth, I now regret my decision. Judging by what he told me on the context of his study, it was certainly his most profound and perhaps most important work [ . . . ] As it happened, the work has not been edited at all, and what is worse nobody seems to know where the manuscript is. [R. Ingarden, *Edith Stein on her activity as an assistant of Edmund Husserl*, *Philosophy and Phenomenological Research*, vol. 23 (1962), pp. 155–175, p. 157n.4]

Apparently unbeknown to Ingarden, after his declining Husserl's proposition, the task was accepted by Husserl's assistant Eugen Fink. However, Fink hardly worked on it and in 1969 he gave the manuscript to the Husserl Archive in Leuven. It was published only in 2001 [Husserl, *Die Bernauer Manuskripte über das Zeitbewusstsein*, *Husserliana*, vol. XXXIII, Kluwer, Dordrecht, 2001]. See footnote 51 of *On the philosophical development of Kurt Gödel*, Mark van Atten and Juliette Kennedy, *The Bulletin of Symbolic Logic* Volume 9, Number 4, Dec. 2003 (ed.).

these wrong concepts consistently.) Time is any order, but the question is what kind of order.

A child begins to form his ideas about time around 4, but only begins to reason between 10 and 12. It's thus not surprising criticism would be needed.

Husserl's epoché on the other hand is essentially an exclusion of criticism, of any concern about truth and falsehood. He even seems to say that everything *is* true. His analyses of the objective world (e.g. p. 212 of "From Formal to Transcendental Logic") is in actuality universal subjectivism, and is *not* the right analysis of objective existence. It is rather an analysis of the natural way of thinking about objective existence.

Following Husserl's program with diligence could lead one finally to a grasping of the system of primitive terms (although there are other ways and perhaps quicker ways). "Die Krisis der europäischen Wissenschaften und die transzendente Phänomenologie" (translated into English) is a work in which Husserl gives indications as to which order one should go about doing phenomenology in. There are also some detailed phenomenological analyses in the Logical Investigations, which were made, however, before 1909. There is an investigations of Objects in Space in one of the latest volumes of Husserliana.

– Husserl's complicated writing style seems to be designed to force you to focus on a certain thing.

**Krisis.** Aim:<sup>2</sup> p. 213 & 14 epoché a new dimension.

137 p. 140 epoché with a religious conversion.

In other sciences, math learn certain facts (conceptual or not).

Phen. completely different.

Aims at insight.

– Whole world appears in a different light.

– Schelling

Plato — Really understand concept of good transcendental insight.

120 p. 121 — wants to lead readers same way he was going.

Really understand *every word*; complicated sentences to slow you down.

p. 240 analogous to psychoanalysis.

p. 155ff order on self-analyses of own cognition.

Read H from time to time directs your attention to certain thing.

Become very clever wrt to one's experiences.

Almost physiological effect.

Epoché is middle between existence & non-existence of outer world.

Proof Phil. & Phen. Res of []<sup>3</sup> of vol 23 1962 p155 manusc on time.

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<sup>2</sup>From this point on the notes have paragraphs that start with a lower case letter and end without a period. The editor has capitalized the first letter of each paragraph and added a period at the end for easier reading (ed.).

<sup>3</sup>Unclear word (ed.).

Also “theorems” in Husserl: no. is small compared to amount he writes.  
They’re not important.

An adequate proof-theoretic characterization of an *idealized* intuitive evidence (this conception being obtained by giving up this restriction to things *we* can understand) will comprise inferences that are not intuitive *for us* & which certainly allow a reduction of the induction inferences to an essentially smaller ordinal.

Kreisel

Elmar

Bishop

What is “same thing” that holds.

Very little connection between way science develops.

He wants something much deeper.

To see what the real reasons for your beliefs are.

As a child, you had these reasons before you; only interested in the consequences you could draw about the new world.

More like an observational science.

**Conversation June 13, 1974.** Read Hilbert’s papers of 1928 — see Church’s bibliography.

What is fundamental in finitism is that things must be able to be *given*, not that finite collections are being dealt with. The restriction to finite collections follows from the fundamental one, for infinite sets cannot be given.

Use the word “finitary”, not “finite” for the kind of reasoning.

Hilbert’s program was completely refuted, but not by Gödel’s results alone. That Hilbert’s goal was impossible became clear after Gentzen’s method of extending finitary mathematics to its utmost limits.

Left out any comments on footnote 2, p. 281 of *Dialectica*. One could consider an idealized finitary mathematician, one who could survey completely any finitary process, no matter how complicated. In this case one *might* be able to obtain an adequate characterization of finitary mathematics. (This is unsolved at the present.) This of course would not help with Hilbert’s program, where we have to use the means at our disposal.

In 2nd Hilbert paper, for instance, the problem that number theory is complete is stated—of course this is completely refuted by Gödel 1931.

**Plato paper.** Orthodox *authority* the question rather than experts.

E. a friend of S., says he is the heart of the state.

*Both* are enemies of orthodox authority. M. is a representative of orthodoxy.

E’s father & S. cannot be the same because E’s father did do something wrong, S. did not.

What is god is what is pleasing to authorities = orthodox church.

E. is half-hearted in his opposition to the orthodoxy, thus no real danger.

### Meaning of the dialogue:

Why the friends of S. didn't prevent his execution — they were half-hearted, didn't take a firm stand against the authorities for S.

- Their shortcomings expressed in the dialogue.
- Joking hint of E. at S's trial.

E's wishy-washy position, never thinking anything through is the cause of him remaining in what is *really* an authoritarian position.

His running off & not carrying through his case is example of his half-heartedness.

Dialogue about the opposition of orthodox & rational religion.

E. prosecuting his father, who had done a criminally negligent thing part of rational religion; orthodox religion emphasized family ties.

**Conversation July 26, 1974.** Mention that Hilbert had posed the problem of the decidability of number theory.

Re comment about Gentzen last time:

The only method of extending finitism is by admitting more recursions. But if we look at the proof of induction up to  $\varepsilon_0$ , we see it is *the* proof of it. And it is an impredicative proof. Even the new proof by Bernays in the new edition of Hilbert & Bernays, which looks finitary, uses Heyting implications, even applied to statements which already contain the implication.

It is said the formalists have interpolated “constructive” means between Brouwer & the finitists. This seems to be true, and that this something is non-finitary. But still impredicative. It might be possible to reformulate Gentzen's work so that it fit into this category.

We can try to see how far we can get finitistically in “seeing” transfinite induction. Certainly we can get to  $\omega^2$ , perhaps even to  $\omega^\omega$ . This may differ from individual to individual, or depend on training. But Hilbert wanted a proof for everyone, not just for those with special training.

We would like to know about the idealized case.  $\varepsilon_0$  might be finitistic in this case.

(Notice that the following are examples of abstract concepts that would have to be avoided for finitary proofs: implication, demonstration, the class of ordinals for which we can apply induction constructively.)

To see (the idealized case?) we must introduce abstract concepts. — Concrete objects in space, concrete relations among them.

(Why do we know the only way of extending finitism is with higher recursions?)

– This is the only way we know for proving general concepts. We can't exclude someone coming up with a new way until we have a complete theory of the mind, but that is a long way off.

**Conversation August 21, 1974.** Corrections to Hilbert paper.

Question: Did Gentzen, in his second paper, still consider it an undecided question as to whether there was a finitary consistency proof for first order arithmetic.

For Hilbert, at  $\omega^\omega$ , induction would still be finitary.

Mathematically, ordinal logics present the math fundamental problems of proof theory.

Turing showed that at a very small ordinal one can decide every mathematical question.

Want to take recursive ordinals given by some canonical definition (cf. Feferman's work; Schütte continued—beyond previously known; definition wouldn't be constructive (would be assigning to each recursive ordinal a definite well-ordering of the integers as has been done for the small ones)

Question then is can you solve every mathematical question in the logics of these ordinals

This is a counterpart of the incompleteness question.

On measuring proof-theoretic strength by the ordinal of its consistency proof: if you allowed your well-ordering to be sufficiently wild,  $\omega^2$  could be used for any system.

Myhill showed it isn't possible to assign ordinals to recursive functions to measure their complexity.

If admit arbitrary well-orderings.

Small ones,  $\varepsilon_0$  etc. are.

To find comment on Brouwer look his name up in the name index of Hilbert's collected works.

An unanswered question is: What is really completely convincing in mathematics. And, can mathematics be reduced to something completely convincing.

$\varepsilon_0$  is just as convincing as finitism.

A neg. answer to this can't be given. Although Gödel thinks not, so far there is no convincing proof.

Would be worthwhile to translate Hilbert-Bernays.

**Conversation Phone Nov 1.** Turing-Feferman result nonsense. Have to be able to reach the logic in some way.

Turing himself looked at it as a negative result.

Feferman has tried to consider properties the canonical representations would have.

Ordinal logics same as adjoining to number theory axioms that these distinguished well-orderings are well-orderings.

Gödel feels can get *completeness* for ordinal logics with canonical well-orderings.

**Conversation July 22, 1975.** Material of Vol XI of *Husserliana* (passive constitution) should have been interesting but doesn't appear to be so.

Work published during Husserl's lifetime appears more interesting.

About ordinal logics: completeness result cannot be constructive.

Rationality of universe

↓

Every set is ordinal definable.<sup>4</sup>

Set, fn, relation, structure.

This *concept* when formed in generality runs into difficulties.

Things other than ordinals which are regular?

Argument of Gödel.

Unreasonable things can make sets ordinal definable.

**Gödel on Intuitionism.** Important to get right idealization to get ideal of finite proof.

Yes — induction principle because not very complex.

May be similar (finitarily valid) principles that we can't recognize because too complicated.

Couldn't recognize them [arrow to "complicated" in previous line] by finitary means but might use more abstract means to show that they are actually finitary recipes.

E.g. double induction.

Use abstract notion of fn. in its formulation.

Intuitionism perfectly meaningful.

In class. math hunt for axioms using extra-mathematical ideas.

But axioms are about mathematical objects.

In intuitionism isn't. Statements involve extra-math. element. Namely, the mind of the mathematician & his ego.

|  
ratl

|  
limits

Statement[s] of int[uitionism] are psychol. statements, but not of empirical psy[chology], — essential a priori psychology/not formal

Formal [vs] other.

|  
math, logic

Restricted kind of evidence: arises from finiteness of ego.

Meaning must be completely *within* the ego.

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<sup>4</sup>A note in the left margin at the implication reads "Gödel  $\rightarrow$ " (ed.).

Conventionalism is an attempt to reduce to the ego alone & to arbitrary decisions of the ego.

Everything is true by (my) convention.

Properties of ink was.

Properties of combining symbols.

It can't reduce arrangement to physical objects.

Problem in intuitions — where classical math. seems to have found its primitive elements. In intuitionist math working with ideas that haven't been analysed (eg concept of proof).

Problems of paradoxes not really important for sets but rather for concepts, truth.

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## STANLEY TENNENBAUM'S SOCRATES

CURTIS FRANKS

Plato was not present on the day that Socrates drank hemlock in the jail at Athens and died. Phædo, who was, later related that day's conversation to Echecrates in the presence of a gathering of Pythagorean philosophers at Phlius. Once again, Plato was not around to hear what was said. Yet he wrote a dialog, "Phædo," dramatizing Phædo's retelling of the occasion of Socrates' final words and death. In it, Plato presents to us Phædo and Echecrates' conversation, though what these two actually said he didn't hear. In Plato's account of that conversation, Phædo describes to Echecrates Socrates' conversation with the Thebian Pythagoreans, Simmias and Cebes, though by his own account he only witnessed that conversation and refrained from contributing to it. Plato even has Phædo explain his absence: "Plato," he tells Echecrates, "I believe, was ill."

We look to Socrates' death from a distance. Not only by time, but by this doubly embedded narrative, we feel removed from the event. But this same distance draws us close to Socrates' thought. Neither Simmias nor Cebes understood Socrates' words as well as Phædo did by the time he was asked to repeat them. Even Phædo failed to notice crucial details that Plato points out. Had we overheard Socrates' conversation, we would not have understood it. We look to Socrates' death from a distance, but to understand Socrates, we don't need to access him—we need Plato.

At Stanley Tennenbaum's instigation, Kurt Gödel and Sue Toledo held a series of conversations between March of 1972 and July of 1975. Gödel retired from his permanent position at the Princeton Institute of Advanced Studies the following year and died shortly afterwards. These conversations were among Gödel's last. Toledo's notes of them are fragmented, often cryptic. Rarely are matters pursued at length. Occasionally, Toledo records merely that a theme or a theorem came up and not what was said about it.

One is struck initially by the wide range of topics one finds in these brief notes, from recent results in mathematical logic to the meaning of ancient texts and the subtleties of modern philosophical thought. But more striking is the way that Gödel's discussions of these disparate topics elide into and

support one another, even spanning breaks of several months. Tennenbaum called himself a “disciple” of Gödel. He saw mathematics as a testament to the dignity of the human mind, and he saw in philosophy a way to reflect on the proper cultivation of that dignity, mathematically and otherwise. Both of these visions, he claimed, he acquired from Gödel. They are vivid still, here, in a form we owe to Tennenbaum, among Gödel’s final sustained thoughts.

“Phædo”’s classical readers referred to it by the alternate title, “On the Soul.” Phædo appears only in the framing narrative. He is silent while Simmias and Cebes ask Socrates what he means when he says that the poet, Evenus, if he is wise, should follow him as soon as possible in death. He only listens as Socrates explains that his imminent death is not an evil and argues that the soul continues to exist after the body dies. Socrates is discussing the soul. Phædo is in the background.

It is not surprising that Socrates’ reasoning is unconvincing to contemporary readers (among the several unpalatable ideas on display is the infamous doctrine that all knowledge is recollection of what was forgotten at birth). The strange thing about the dialog is that Socrates has an unusually hard time persuading his own friends. Plato devotes roughly twenty-five pages to Socrates’ attempt to demonstrate the immortality of the soul. The progression is repetitive, cyclic. Simmias and Cebes object continuously. When one of them accepts one of Socrates’ points, the other typically does not. Socrates changes his argument a few times. Usually he tries to demonstrate that the soul is immortal by its nature. Occasionally (80e, 81b, 82c) he argues for the seemingly contrary claim that only the souls of philosophers outlive their bodies, because immortality is conditional on one’s conduct in this life. When neither of these tactics proves fully convincing, he rehashes some of the images and analogies from the beginning of the discussion in a slightly different order. The result is particularly elegant, but ultimately unpersuasive: Simmias finds more compelling the notion that the soul depends on the body because it is properly understood as a harmonious arrangement of the bodily; Cebes disagrees with Simmias on this point but finds Socrates’ metaphor about the cyclic relationship between life and death untenable.

At this point in the dialog, Phædo interrupts the flashback and says to Echecrates:

When we heard what they said, we were all quite depressed, as we told each other afterwards. We had been quite convinced by [Socrates’] previous argument, and [Simmias and Cebes] seemed to confuse us again, and to drive us to doubt not only what had already been said but also what was going to be said, lest we be worthless as critics or the subject itself admitted of no certainty. (88c)

Echecrates replies that even hearing of the discussion after the fact, he feels that he shares in their despair. Socrates' argument, he says, "was extremely convincing" but has "fallen into discredit." He pleads that Phædo relate Socrates' response precisely, as he is interested both in what new argument Socrates devised at this point and in whether he remained composed in the face of these objections. Phædo says:

I have certainly often admired Socrates, Echecrates, but never more than on this occasion. That he had a reply was perhaps not strange. What I wondered at most in him was the pleasant, kind, and admiring way he received the young men's argument, and how sharply he was aware of the effect the discussion had on us . . . (88e–89a)

What did Socrates say that impressed Phædo more than any clever argument he had ever presented? Setting aside Simmias and Cebes' objections, he says, "first there is a certain experience we must be careful to avoid" (89c). Only here does Phædo enter into conversation with Socrates. "What is that?" he asks. "That we should not become misologues," Socrates replies, "as people become misanthropes. There is no greater evil one can suffer than to hate reasonable discourse" (89d). He elaborates:

You know how those in particular who spend their time studying contradiction in the end believe themselves to have become very wise and that they alone have understood that there is no soundness or reliability in any object or in any argument, but that all that exists simply fluctuates up and down as if it were in the Euripus and does not remain in the same place for any time at all. . . . it would be pitiable . . . if a man who dealt with such arguments as appear at one time true, at another time untrue, should not blame himself or his own lack of skill but, because of his distress, in the end gladly shift the blame away from himself to the arguments, and spend the rest of his life hating and reviling reasoned discussion . . . . (90b–d)

Socrates eventually proceeds with his investigation of the immortality of the soul. Around 107a–b, the discussion winds down. Socrates has introduced a few novelties in the argument. Largely, though, he preserves substantial patches of what he said earlier. Cebes announces that he is convinced. Simmias is more cautious: "I myself have no remaining grounds for doubt after what has been said; nevertheless, in view of the importance of our subject and my low opinion of human weakness, I am bound still to have some private misgivings about what we have said." Astonishingly, Socrates endorses Simmias' subtle reservation:

You are not only right to say this, Simmias, . . . but our first hypotheses require clearer examination, even though we find them convincing. And if you analyze them adequately, you will, I think,

follow the argument as far as man can, and if the conclusion is clear, you will look no further.

Thus unfolds Plato's treatise on the soul. Socrates' proof falls just short, but he proceeds to hypothesize about the nature of the existence of disembodied souls and adds that

[n]o sensible man would insist that these things are as I have described them, but I think it is fitting for a man to risk the belief—for the risk is a noble one—that this, or something like this, is true about our souls and their dwelling places, since the soul is evidently immortal, and a man should repeat this to himself as if it were an incantation, which is why I have been prolonging my tale. (114d)

Still speaking, he takes the cup of poison, "and then drain[s] it calmly and easily" (117c). Crito weeps, but Socrates speaks on and everyone gains their composure in time to witness his death.

Socrates' friends managed to reconcile themselves to his passing. His death was not a bad thing. Socrates' final words brought them to this understanding, but the conclusion he emphasized was not that the soul is immortal. When Phædo recounted this episode to Echechrates, he said that Socrates' intense insistence that reasoned discourse is noble even when not entirely satisfying was his greatest teaching. It allowed them not to succumb to doubt "about what was going to be said" as they had come to doubt "what had already been said." Socrates even was eager to point out explicitly that his argument was in some respects weak, that although they had taken it as far as possible under the conditions, it would benefit still if its initial hypotheses were further clarified at a later time. Socrates' proof that the soul is eternal convinced nearly everyone; his admission that he was not entirely satisfied with it fixed the soul's immortality in his friends' minds like an incantation.

The first block of Toledo/Gödel notes is dated March 3, 1972. The conversation from that day is idiosyncratic in that its only topic, Edmund Husserl—his thought but also his person—does not recur on any later date<sup>1</sup>. And yet the centrality of Husserl in Gödel's thinking is unmistakable here. Gödel and Toledo don't talk about anything else, and in none of the later conversations do they discuss anything with so much care.

Many readers will be surprised to find that Gödel says essentially nothing about that part of Husserl's work most often associated with formal logic, his early writings on logical form and content including *Logical Investigations*. He is interested in a properly phenomenological subject, Husserl's emphasis on *epoché* following a transformative philosophical "discovery" in 1909. In fact,

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<sup>1</sup>There is one exception: The notes from July 22, 1975 begin with the observation that the content of some of Husserl's unpublished manuscripts, which had recently been made available, was uninteresting compared with his published work.

the discovery itself, its nature, interests Gödel even more than the philosophical mode that Husserl adopted in its wake. He calls it the moment of “grasping the system of primitive terms and their relationships.” This moment is not unique to phenomenology—Gödel identifies the same transformation in Descartes, Schelling, Leibniz, and Plato. He calls it the definitive moment in the “life” of every “real philosopher.” Whatever it amounts to, it defines a thinker’s *life*, and not only his or her *thought*. Husserl, Gödel notes, underwent “both intellectual and personal” crises in 1909, and his transformative discovery resolved both.

Toledo and Gödel discuss Husserl attentively. They are interested in the nuance of how his thought progressed more than the content of what he says on any particular occasion. They see Husserl inviting us to come along with him, showing us how to transform ourselves in the most fundamental ways. After 1909, Husserl stopped producing “fully worked out phenomenological investigations.” He introduced the stylistic convention of deliberately complicated language that forces us to think about each of his words. He wanted us to “use his experience to get to” where he was. Husserl had attained a kind of understanding that “cannot be transferred from one person to another,” so his writing is not an attempt to explain anything to us. Gödel calls it both the understanding of the primitive terms and the understanding of “the absolute.” We are blind to this absolute. We are not aware of how we are working with our primitive terms, so we are bound to distort any image of the absolute that Husserl could present. Husserl must force us to notice our own primitive terms, to diagnose our own ways understanding, “to see what the real reasons for [our] beliefs are.” The transformation that accompanies this self-awareness is not conceptual. To emphasize the impossibility of conveying it, Gödel compares it to religious conversion and calls it “almost physiological”—reports that suggest that Gödel believes he himself has experienced it.

Gödel focuses on this philosophical strategy. It sounds like a first step in a larger program: after completing what Gödel calls our “self-analysis of [our] own cognition,” we expect to move on to the adoption of the right conceptual grid, to take off the glasses that have distorted everything and try on the new set that Husserl is offering. But if Husserl was interested in this second phase, Gödel is not. The self-analysis, what Gödel says is analogous to psychoanalysis, is the crucial move. We are not aiming at the acquisition of any new facts, not even of “conceptual” facts (There are very few “theorems” in Husserl’s writing, Gödel says). We want to see the “whole world . . . in a different light”—what Gödel calls “insight.” To do this, we don’t need to adopt new primitive terms, we only need to recognize the ones that we are working with and to become aware of how we are working with them. Gödel doesn’t seem committed even to the *possibility* of changing our primitive terms. We want simply to find them, to make them clear. This means that we should

become aware of how it is that we use them, how we *in fact* handle them as opposed to how we ought to do so.

This is the key to understanding a potentially disorienting part of Toledo's notes. Gödel says that after 1909 Husserl's thinking was Kantian, that his philosophical method was "critical" in the sense associated with Kant's transcendental idealism. The significance of this is that our primitive terms could be anomalous, out of sync with the noumenal world, moreover that we might not "handle them correctly" in any case. Indeed, it is highly unlikely that our primitive terms are "the right ones" in any deep sense and practically certain that our handling of them is flawed from a transcendental point of view ("it's not surprising criticism would be needed," Gödel says, given the fact that we form our ideas in childhood well before the way we work with them can meaningfully be called thinking). The term "*kritik*" is supposed to suggest this: our basic ways of thinking are subject to criticism. But in a statement that is somewhat jarring, Gödel says that he admires the fact that Husserl does *not* supply the criticism that is doubtless warranted. "There is no reason to assume that we always handle [these concepts] correctly," but notice, Gödel says, that Husserl didn't mention this. His *epoché* "is essentially an exclusion of criticism, of any concern about truth and falsehood." We are not supposed to understand the quest for primitives as a quest for the fundamental features of reality, but "rather as an analysis of the natural way of thinking about" that reality. We don't want to evaluate that way of thinking—and *epoché* ensures that we can't—but to become aware of it because we can't gain insight about our world or our activities without awareness of how it is that we think about the world. Prior to Husserl's transformation, he might still have considered the possibility that our deployment of primitive terms tracks the form of the world, or at least that it could be made to do so. After his transformation, he realized that assumptions of this sort are careless. But he went even further by insisting that insight is to be attained not by correcting this "problem" but by ignoring it and learning to be concerned about something else.

Gödel criticizes "analytic" philosophy for its insensitivity to this sort of insight. Its practitioners "try to make concepts clear by defining them in terms of primitive terms," but, Gödel says, they don't bother "to make the primitive terms clear," by which he means that they don't bother to first get clear about how we handle those primitive terms. "Moreover," Gödel says, "they take the wrong primitive terms." What kind of criticism is this? It might seem that Gödel is pointing out that analytic philosophers look at the world from the wrong angle, that this is their shortcoming. But this is not what Gödel is saying. He thinks they err by deceiving themselves about how they in fact look at the world. What they take to be primitive terms, the terms in which they try to define all of our concepts, are not actually the fundamental components of our thinking. Perhaps they ought to be, but they

are not. Husserl's understanding of the absolute came with the realization that dissecting concepts according to how they ought to be seen is a distraction from the sort of insight that can transform our thinking and resolve our personal crises. Analytic philosophers err by working with a false self-image, or by thinking that self-understanding is irrelevant to conceptual analysis. So they analyze everything in terms of concepts that are not fundamental for us and are left with no insight into our concepts.

So far I have said very little about the argument in "Phædo" itself, but one detail is pertinent. This is the hypothesis of opposites, a crucial instance of which incited Cebes' initial objections. Socrates poses it thus:

[F]or all things which come to be, let us see whether they come to be in this way, that is, from their opposites if they have such, as the beautiful is the opposite of the ugly and the just of the unjust and a thousand other things of the kind. Let us examine whether those that have an opposite must necessarily come to be from their opposite and from nowhere else, as for example, when something comes to be larger it must necessarily become larger from having been smaller before. (70e)

The hypothesis is revived against Cebes' objections and takes on a central (if occasionally cryptic) role in Socrates' discussion throughout the dialog. But where did it come from?

Phædo's account of Socrates' last day begins early in the morning. The Athenian jail guard doesn't let Socrates' friends in initially. When eventually they do enter, Socrates has recently been released from his shackles. Still lying in bed, Socrates addresses Crito. Then he sits upright, "ben[ds] his leg, and rub[s] it with his hand," saying as he does:

What a strange thing that which men call pleasure seems to be, and how astonishing the relation it has with what is thought to be its opposite, namely pain! A man cannot have both at the same time. Yet if he pursues and catches the one, he is almost always bound to catch the other also, like two creatures with one head. I think that if Æsop had noted this he would have composed a fable that a god wished to reconcile their opposition but could not do so, so he joined their heads together, and therefore when a man has the one, the other follows later. This seems to be happening to me. My bonds caused pain in my leg, and now pleasure seems to be following. (60b-c)

We learn immediately after this passage that Socrates has spent his time writing poetry—specifically transposing Æsop into verse—since his incarceration. This activity apparently has played into his current observation, or at least into its preliminary formulation as a principle. When Socrates later turns to

discuss the immortality of the soul, he reaches for this principle—presumably not because it is evidently connected with the statement he is trying to prove, but for the plain reason that it has recently been on his mind. He has discussed the soul's immortality in the past, but no one present can recall the arguments from those occasions (73a, 76b, 88c). The "theorem," as Gödel would call it, is known to Socrates, but he hasn't yet found the memorable proof. He has been reading *Æsop*. He recently saw a familiar experience in a new light. He tries another angle.

There is no record of a conversation between Toledo and Gödel for two years following the Husserl discussion. The next set of notes, dated June 13, 1974, is split between one discussion of David Hilbert's papers from 1928 and another one of Plato's dialog, "Euthyphro." The two discussions are not evidently related. The comments on Hilbert's foundational program are briefer, but thematically connected with the material in the four conversations to follow. The slightly more extended discussion of "Euthyphro," by contrast, seems initially out of place in the broader context.

Surely the most striking remark recorded here is that Hilbert's program "was completely refuted," though by Gerhard Gentzen's work and not by Gödel's own results. This is exactly opposite the customary appraisal of these matters. Gödel's results from 1931 are widely recognized as a refutation of Hilbert's project, for they show that the consistency of certain precisely delimited mathematical theories cannot be proved using only those same theories' means, whereas Hilbert had sought "finitary" proofs of the reliability of abstract, infinitary mathematical techniques. Notoriously, in his 1931 paper, Gödel cautioned against drawing this conclusion. Maybe there are perfectly concrete principles of inference, he suggested, of the sort that Hilbert would countenance, that surpass the techniques present in any formal system and that suffice to prove any such system's consistency.

In 1936, Gentzen presented his work on arithmetical consistency in this light: he had shown how to supplement manifestly "finitary" techniques with principles of transfinite induction so that these combined resources suffice to prove the consistency of formal arithmetical theories. One need only take induction principles through sufficiently high ordinals according to the complexity of the theory one is investigating. Can't such consistency proofs be counted as finitary? "We might be inclined to doubt the finitist character of the 'transfinite' induction [through  $\varepsilon_0$  used in his proof of the consistency of Peano Arithmetic (PA)]," he wrote in *Gentzen 1938*,

even if only because of its suspect name. In its defense it should here merely be pointed out that most somehow constructively oriented authors place special emphasis on building up constructively ... an initial segment of the transfinite number sequence ... . And

in the consistency proof, and in possible future extensions of it [to theories stronger than PA], we are dealing only with an initial part, a “segment” of the second number class . . . . I fail to see . . . at what “point” that which is constructively indisputable is supposed to end, and where a further extension of transfinite induction is therefore thought to become disputable. I think, rather, that the reliability of the transfinite numbers required for the consistency proof compares with that of the first initial segments, say up to  $\omega^2$ , in the same way as the reliability of a numerical calculation extending over a hundred pages with that of a calculation of a few lines: it is merely a considerably *vaster* undertaking to convince oneself of this certainty . . . . (p. 286)

Yet Gödel suggests just the opposite, that—far from being a way around the implications of his incompleteness theorems—Gentzen’s work fully refuted Hilbert’s program. Why?

In the notes from July 6 of the same year, Gödel says: “We can try to see how far we can get finitistically in ‘seeing’ transfinite induction. Certainly we can get to  $\omega^2$ , perhaps even to  $\omega^\omega$ . This may differ from individual to individual, or depending on training. But Hilbert wanted a proof for everyone, not just for those with special training.” This is a novel idea in the evaluation of Hilbert’s program<sup>2</sup>. Gentzen had said that it is not evident at what point principles of transfinite induction lose their constructive nature. Gödel replies that this is irrelevant. “One could consider an idealized finitary mathematician, one who could consider completely any finitary process, no matter how complicated. In this case, one *might* be able to obtain an adequate characterization of finitary mathematics,” he says. “We would like to know about this idealized case.  $\varepsilon_0$  might be finitistic in this case.” But for us even to see the idealized case, “we must introduce abstract concepts.” Thus “this is no help for Hilbert’s program,” he says, “where we have to use the means at our disposal.”

The notes from August 21, 1974 show Gödel approaching the same issue from a different angle. In place of the question “Where in the progression of transfinite ordinals do things lose their finitary nature?” Gödel considers the question of the naturalness of primitive recursive well-orders used to define elementary order-types and notes that “if you allowed your well-ordering to be sufficiently wild,  $\omega^2$  could be used for any system” (The force of this observation is compounded by Gödel’s claim that, “for Hilbert, at  $\omega^\omega$  induction would still be finitary.”). Gödel here is referring to the appeal to meta-mathematical notions in definitions of small order-types. A concrete example will illustrate the point (the example is due to Kreisel): First define a primitive recursive predicate  $P(x) \leftrightarrow \exists y \leq x \text{Prf}_{PA}(\ulcorner \perp \urcorner, y)$ . Then define a binary relation  $\preceq$

<sup>2</sup>In chapter 2 of *Franks 2009* a case is made that Hilbert ought to be understood in this way, as insisting on a “proof for everyone.”

as follows:

$$\begin{array}{lll}
 2n \preceq 2m & \text{iff} & n \leq m \\
 2n \preceq 2k + 1 & \text{iff} & \neg P(n) \wedge P(k) \\
 2k + 1 \preceq 2m & \text{iff} & P(k) \wedge P(m) \\
 2k + 1 \preceq 2l + 1 & \text{iff} & P(k) \wedge P(l) \wedge l \leq k
 \end{array}$$

Observe that  $\preceq$  has order-type  $\omega$  if PA is consistent and contains a finite sequence bounded by a strictly decreasing infinite sequence otherwise. It is fairly straightforward<sup>3</sup> to prove the consistency of PA by an elementary induction on  $\preceq$ , hence, since PA is consistent, on a recursive well-order of type  $\omega$ . However, this well-order is unnatural (Gödel calls it “wild”)—we would never be able to define  $\omega$  in this way without appealing to our intuition that PA is consistent.

So how ought we distinguish natural definitions from unnatural ones? Not, Gödel thinks, by an analysis of the distinction between meta-mathematics and ordinary mathematics or any other such distinction, but by *self-analysis*, by getting clear about our own finitary constructions rather than investigating the ideal case. Thanks to the juxtaposition of these remarks with those from two years earlier, the influence of Husserl on Gödel’s evaluation of Hilbert’s program is unmistakable. The essential link appears in the course of the discussion of *kritik* from March 3, 1972:

An adequate proof-theoretic characterization of an idealized intuitive evidence (this conception being obtained by giving up this restriction to things *we* can understand) will comprise inferences that are not intuitive *for us* & which certainly allow a reduction of the inductive inferences to an essentially smaller ordinal.

In the remarks from August 21, 1974, Gödel asks whether “Gentzen, in his second paper, still considered it an undecided question as to whether there was a finitary consistency proof for first order arithmetic.” Gödel is referring to *Gentzen 1938*, the crucial passage of which is quoted above. (In Gentzen’s “first paper” *Gentzen 1936*, he argued more forcefully for the claim that his consistency proof was finitarily acceptable. In this “second paper” Gentzen expresses dissatisfaction with that argument, but still clearly suggests that the question is open.) Gödel’s question seems inexcusably cautious until it is understood in this way: Gentzen clearly considered this question undecided in 1938, but he was referring to proofs that were “essentially” finitary as opposed to proofs that were finitary *for us*. This is what Gödel means in the passage from June 13, 1974, when he says, “what is fundamental in finitism is that things must be able to be given, not that finite collections are being dealt with.” Gödel wonders whether Gentzen could seriously understand foundational programs in terms of such an idealized grounding of mathematical activity,

<sup>3</sup>One may consult §7.1.9 of *Girard 1987* for the proof and a general discussion.

as opposed to a more anthropomorphic grounding in our actual primitive terms. As for the latter conception of mathematical foundations, Gödel sees Gentzen's work, rather ironically, as being quite decisive. In the notes from July 26 of that year, Toledo wrote: "[I]f we look at [Gentzen's] proof of induction up to  $\varepsilon_0$ , we see that it is *the* proof of it. And it is an impredicative proof . . . , which [perhaps from some angle] looks finitary," but implicitly appeals to abstract notions. Gödel means that once we understand this proof, we recognize both that it is the natural proof and that it exceeds our actual ability to work with things given to us in intuition. "That Hilbert's goal was impossible became clear after Gentzen's method of extending finitary mathematics to its utmost limits," he proceeds to say, because this method made evident that the natural proof of PA's consistency exceeded those limits.

In the dialog "Euthyphro," Plato dramatizes a possibly fictitious encounter between Socrates and a priest named Euthyphro. The two meet outside the king-archon's court, where alleged affronts to the Olympian gods are adjudicated. It is clear that Euthyphro and Socrates know one another, though their exact relationship is left vague.

Euthyphro is surprised to find Socrates here, given his habit of remaining aloof from civil matters. Socrates explains that he is not here on his own initiative but has been charged by a man named Meletus with harming Athens by spreading heretical ideas. (These are the charges on which Socrates will soon be convicted, imprisoned, and eventually executed.) He has come to the court for the preliminary stages of his hearing. Euthyphro is aghast. Socrates, he says, far from posing a danger to the state, is "the very heart of the city" (3a).

Euthyphro's presence outside the court is, by contrast, unremarkable: his religious station makes his testimony relevant to court procedures. All the same, Socrates asks him about his current court business. Euthyphro explains that he is pressing charges of murder on his father because of an episode of negligence that resulted in the death of a former house servant. When Socrates points out that it is considered scandalous for a son to prosecute his own father, Euthyphro replies that he is confident that public opinion is wrong on this point, that he is sufficiently "advanced in wisdom" to see past societal conventions to the fact that justice demands equitable treatment of relatives and strangers (4b).

Socrates' interest is piqued. In an ironic plea that Euthyphro takes seriously, he says that because Euthyphro is such an expert in divine affairs he would like to enlist under his tutelage so that the charges being brought against him might be deflected to his new official adviser in spiritual matters. Euthyphro agrees to this arrangement without hesitation. Socrates insists that they begin right away and asks Euthyphro to explain to him the nature of *hosion* ( $\delta\sigma\iota\omega\nu$  = piety, holiness, sacred matters).

Euthyphro first tries to illustrate *hosion* by pointing to his own plan to prosecute his father as an example: "You want to know about piety? I'll show you piety. Watch what I'm doing, these circumstances. Learn piety by seeing a pious person in action!" Socrates says that this isn't how he expects to learn things and demands instead a "formal" definition. Euthyphro, seemingly reluctantly, agrees to try this out: "If that is how you want it, Socrates, that is how I will tell you" (7e). He first tries to define *hosion* as "what is loved by the gods." Socrates finds a problem with that definition, in that the gods of the day had competing loves. So Euthyphro amends his definition slightly by saying that *hosion* is what all the gods agree to love. This leads into the famous causal dilemma: Does the gods' mutual love of a thing make it pious, or does the piety of a thing earn the gods' love?

This dilemma is easily associated with Plato's alleged ontological doctrine that an abstract phenomenon's nature is uninfluenced by an agents' knowledge of or decisions about it. Euthyphro's attempt to define *hosion* in terms of the gods' love doesn't do justice to its alleged eternal "form." If whatever the gods agree upon as pleasant thereby becomes holy, then indeed holiness does not *have* an eternal form. It must rather be, Socrates argues, that a thing's holiness attracts the gods' love of it. If so, then defining *hosion* as "what the gods love" is like defining it as "what you and I are talking about right now"—one has identified merely an "affect or quality" of holiness, not its essence (11a).

Not equipped to navigate the causal dilemma satisfactorily, Euthyphro proposes a third definition, namely, that *hosion* is part of justice. Socrates presses Euthyphro into specifying which part of justice it is, and Euthyphro says that it is a kind of trading relationship with the gods. We receive from the gods, so we return this favor with certain behaviors. Socrates asks what the gods receive from us through these behaviors, and Euthyphro begins to speak eloquently about the cosmic harmony brought about by a ritually meticulous and duly reverent society at prayer (14b). But Socrates will have none of this kind of talk and cuts him off, insisting again on a "formal" definition. "What benefit," he asks, "do the gods receive from our pious behavior?" Euthyphro replies rhetorically, "Do you suppose, Socrates, that the gods are benefited by what they receive from us?" (15a). Socrates accepts this and then asks what piety's purpose is, if not benefit. How is it repayment for the good that we receive from the gods? Euthyphro says that the display of honor and reverence is pleasing to the gods. Socrates then points out that this claim, understood formally, is just a repetition of their earlier failed attempt to define piety: piety is what pleases the gods. They have covered no ground.

After each of Euthyphro's failures to sufficiently explain *hosion*, Socrates has urged him to collect himself and to try again so that he can become Euthyphro's pupil and absolve himself of the charges he faces. Until now, Euthyphro has been a relatively good sport, trying out new angles despite

Socrates' commanding refutation of all his ideas. But this time Euthyphro runs off, somewhat impatient.

One who would decipher this dialog faces a strange amalgam of earnestness and irony. The charges brought against Socrates are grave, and in "Apology" and "Phædo" it becomes clear that they are not just idle threats. Socrates' reaction to them is befuddling. He says that he would like to deflect the charges onto Euthyphro. Is there really any possibility of doing this? If not, then shockingly, Socrates would appear to be joking at the least appropriate time. On the other hand, if this strategy is viable, then it is equally odd that Socrates would make the arrangement conditional on Euthyphro's demonstration of expertise. Shouldn't the priest's reputation suffice? Moreover, if these terrible charges can so easily be deflected, then it is puzzling why Euthyphro, who evidently is not equipped to define piety in a way that holds up to scrutiny, would agree to the arrangement so unhesitatingly. It's fairly clear that Euthyphro didn't take the exercise seriously to begin with and that Socrates never expected Euthyphro to define anything satisfactorily.

What, then, does Socrates hope to accomplish with these antics as his fateful trial draws near? At 5d and again at 6e Socrates asks Euthyphro whether he agrees that *hosion* "presents us with one form." Both times Euthyphro agrees. In the bulk of the dialog that follows, we see how ill-equipped Euthyphro is at specifying that "form." Socrates expects this display of ineptitude. His purpose is to expose the ignorance behind Euthyphro's pretension to wisdom. What seems like a curious preoccupation on the eve of one's heresy trial is itself, for Socrates, pious behavior—to "go around seeking out anyone, citizen or stranger, whom I think wise . . . [and] if I do not think he is [to] come to the assistance of the god and show him that he is not wise" ("Apology" 23b).

Euthyphro's reception of this treatment is noteworthy. After stumbling over the causal dilemma the first time, he says, "I have no way of telling you what I have in mind, for whatever proposition we put forward goes around and refuses to stay put where we establish it." Socrates replies, ". . . [i]f I were stating them and putting them forward, you would perhaps be making fun of me and say that because of my kinship with [Dædalus] my conclusions in discussion run away and will not stay where one puts them. As these propositions are yours, however, we need some other jest, for they will not stay put for you, as you say yourself." To this Euthyphro says, "I think the same same jest will do for our discussion . . . for I am not the one who makes them go around and not remain in the same place . . . ; for as far as I am concerned they would remain as they were" (11c–d). When, at the end of the dialog, Socrates points out that their discussion has cycled back to the same conundrum, Euthyphro throws up his hands and leaves. Socrates takes note. When the "form" he seeks to disclose proves elusive, Euthyphro seemingly concludes "that there is no soundness or reliability in any object or in any argument, but that all that

exists simply fluctuates up and down as if it were in the Euripus and does not remain in the same place for any time at all.” He doesn’t “blame himself or his own lack of skill but, because of his distress, in the end gladly shift[s] the blame away from himself to the arguments.”

It is well known that Gödel defended a sort of “Platonism” about mathematical truth, a thesis that meaningful mathematical talk is explicable only by there being a mathematical reality whose details don’t depend on our ability to determine them. In *Gödel 1964*, Gödel considered the possibility that “Cantor’s conjecture” (that the continuum has the least possible cardinality greater than the cardinality of the set of integers) might be independent of the standard set-theoretical axioms and remarked that

a proof of the undecidability of Cantor’s conjecture from the accepted axioms of set theory . . . would by no means solve the problem. For if the meanings of the primitive terms of set theory . . . are accepted as sound, it follows that the set-theoretical concepts and theorems describe some well-determined reality, in which Cantor’s conjecture must be either true or false. Hence its undecidability from the axioms being assumed today can only mean that these axioms do not contain a complete description of that reality. (p.260)

In his “Gibbs lecture” he reaffirmed his belief in a mathematical reality independent of human conventions and behaviors: “the objects and theorems of mathematics are as objective and independent of our own free choice and our own creative acts as is the physical world” (*Gödel 1951*, p. 312).

It is thus remarkable that in Gödel’s probing discussion of “Euthyphro” in 1974, he says nothing about the dialog’s ontological implications. Instead, Gödel reads “Euthyphro” as a call to wrestle religiosity and moral conviction from the dictates of authority. He associates the idea that the gods not only command us to do what is right but that their commands actually determine what is right with “orthodox religion.” Orthodoxy, Gödel suggests, confuses expertise with authority. We look to experts for answers and advice with good reason, but institutions take advantage of our tendency to trust experts: they transform our reasonable habit of turning to experts as *our source for facts* into the scandalous idea that their opinions are *the source of truth*. Orthodox religion arises in the wake of this transformation and thus struggles to conceive of a deity other than on this authoritarian model, i.e., as investing particular behaviors with moral worth by commanding their performance. The usual grounds for objecting to this theology is that such gods’ commands can defy reason, and we are therefore expected to defy ourselves insofar as we are expected to heed those commands. Better, a theology where the gods’ commands direct us to patterns of living and thinking that we can rationally appreciate. This is what Gödel calls “rational religion.” He reads the causal

dilemma about the relationship between what pleases the gods and *hosion* as fundamentally an argument between rational and orthodox religious thought.

Gödel sees this very same dilemma appearing a second time in the dialog, this time concerning our duty to the state instead of our duty to the gods. The orthodox view is that whatever the civil authorities deem is *de facto* what you are obligated to do for the state. The rational view is that the civil authorities are beholden to some “exterior” facts about what sort of behavior among its citizens would be good for the state, and are thereby obligated to enforce this, rather than just whatever they please.

According to Gödel, the dialog between Socrates and Euthyphro is about how to break away from the authoritarian view of morality. Both speakers agree that the break is needed. Athenian society has established an implicit taboo on prosecuting one’s own father. Socrates reminds Euthyphro (ironically) of this fact when he hears of his purpose in court, trying to elicit a reaction to the apparently common view that such societal conventions are constitutive of right behavior. Euthyphro passes this test and also another: when he learns that Meletus, the representative of “orthodox authority” has brought charges against Socrates, Euthyphro has a chance to add his voice to that charge but instead accuses Meletus of harming the state.

But Euthyphro and Socrates agree only this far, and Gödel sees the dialog’s significance as coming a step after its characters’ mutual rejection of authoritarian ethics. When one recognizes that important truths are not true by convention—not by our individual whims, nor by communal consensus, nor even the dictates of recognized authority—the obvious task one sets oneself to is devising a method to discover these truths. The Socratic proposal is to resort to reasoned discussion. But how ought one react when our conversation falls obviously short of its target? For Gödel, *everything* depends on how we answer this second question. Euthyphro reacts with frustration. “I was right,” he might say, if he weren’t too impatient to stick around any longer, “to doubt that this would be an effective way to learn anything about piety. Do you honestly think it will be worthwhile to approach this problem again?” Euthyphro can see pretty clearly that a satisfactory definition is beyond their reach. Their initial attempts to produce one have only made the impossibility of the task more evident. Socrates’ willingness to press further appears quixotic. But for Socrates, the realization that an eternal truth is beyond one’s understanding is the *beginning* of inquiry. New, more delicate questions emerge: “What can I learn about my own assumptions from this discovery?” “How can I avoid succumbing to authority at this point?” “What must I do to keep conversation on this topic meaningful, now that I cannot sincerely hope to learn what I was originally interested in?”

How can Euthyphro so glibly agree to Socrates’ request that he stand trial in his stead? Gödel sees this reaction as an act of betrayal, for it makes evident the fact that Euthyphro doesn’t expect his discussion with Socrates to be

sufficiently conclusive for the arrangement to materialize. What Euthyphro fails to realize is that Socrates is not testing him for expertise. The condition Socrates places on entering a cooperative with Euthyphro isn't that the priest first define piety but that, despite his inability to do so, he remain committed in his stand against authoritarianism. Gödel calls it half-heartedness, the commitment to "ontological Platonism" without an accompanying commitment to what one might call "moral Platonism." He faults Euthyphro with letting the fact that the nature of piety defies rationalization keep him from defending Socrates, and reason, in court. Authoritarianism reigns not only when everyone is convinced by it, but also when we despair of standing whole-heartedly against it simply because we realize that we can't conclusively explain what the authorities dishonestly claim they control. Had Euthyphro not backed down and had Socrates' friends joined in the stand against authority instead of trying to devise schemes for Socrates' escape, then, Gödel claims, Socrates' execution would have been prevented.

Thus the significance of Gödel's belief in a "well-determined" mathematical "reality" is not the plain fact that he held this view nor, ultimately, the reasons he gave to support it. What is crucial is that he stressed his ontological Platonism even in the face of systematic incompleteness, and that he was in turn prompted by his moral Platonism to devise new ways to attain a synoptic view of the collective body of mathematical facts rather than to despair of any possibility of doing so. In the conversation of August 21, 1974, he asks whether all mathematical questions can be solved in the logics of transfinite ordinals. Later that year, in the conversation from November 1, he expresses his belief that a sort of completeness result can be obtained for ordinal logics. His own discoveries from half a century earlier could easily dissuade one from pursuing this sort of problem any further, yet in Gödel's hands they seem only to have led to a richer view of the sort of questions that can be asked.

In the last section of the notes from July 22, 1975, Gödel discusses mathematical intuitionism. He claims that mathematics "seems to have found its primitive elements" in intuitions, and he contrasts the classical and the intuitionistic reactions to this discovery. Each school runs blindly with one of the two tenets of Husserl's thought that Gödel appreciates. The classical mathematician "hunts for axioms" using ideas from outside of mathematics. "But," Gödel says, the "axioms" that result "are about mathematical objects." The classical mathematician, then, retains the possibility of "criticizing" his or her methods, of recognizing that they aren't adequate to their subject matter. By contrast, every statement of intuitionism involves reference to "the mind of the mathematician & his ego." The meaning of such a statement "must" therefore "be completely *within* the ego." Thus intuitionism, by disallowing criticism, will more readily accommodate a self-analysis, leading to clarity about our actual intuitions. One must seemingly choose between assuming the critical stance and clarifying our intuitions.

But Gödel advocates the simultaneous cultivation of insight and critical awareness in mathematics. One should neither rest content with “working with ideas that haven’t been fully analysed” nor risk flirting with conventionalism. Both mistakes are carelessly myopic, but Gödel’s distaste for the latter peril is also vividly moral. In the notes from August 21, 1974, we read, “an unanswered question is: What is really convincing in mathematics. And, can mathematics be reduced to something completely convincing? . . . Although Gödel thinks not, so far there is no convincing proof.” In its least critical form, intuitionism is such an attempt at reduction: “Conventionalism is an attempt to reduce to the ego alone & to arbitrary decisions of the ego,” he says the following year. What makes this reduction unconvincing, though, is its moral failing. Toledo writes, “everything is true by (my) convention.” The idea conveyed parenthetically seems to be that a little pressure on such conventionalist lines brings their latent authoritarian connotations to the surface. What makes our primitive terms worth scrutiny, Gödel thought, is not that they are constitutive of anything and therefore beyond reproach, but simply that they are what we have to work with and that we work with them much better when we know something about them. The nobility of the human mind lies not in its role as arbiter in crucial matters, but in its ability somehow to tap into matters that transcend it.

When Phædo arrives in Phlius, Echecrates immediately asks him if he was present at Socrates’ death, as he is anxious to hear about the event. Phædo lets him know that he was there and that he has plenty of time to talk. Echecrates asks for every detail. Phædo prepares to relate the meandering discussion and the dramatic scene. He will start with Socrates’ observation about pleasure and pain, that “a man cannot have both at the same time.” But even before that he wants simply to convey the feeling of the moment:

I certainly found being there an astonishing experience. Although I was witnessing the death of one who was my friend, I had no feeling of pity, for the man appeared happy in both manner and words as he died nobly and without fear . . . . That is why I had no feeling of pity, such as would seem natural in my sorrow, nor indeed of pleasure, as we engaged in philosophical discussion as we were accustomed to do—for our arguments were of that sort—but I had a strange feeling, an unaccustomed mixture of pleasure and pain at the same time as I reflected that he was just about to die. All of us present were affected in much the same way . . . . (58e–59a)

Thus at the crucial moment when Socrates’ words must be effective, just as he is about to die, his friends share a feeling that undermines his argument. Though it led them that day to see their teacher’s death not as a bad thing, the hypothesis of opposites is false. But they don’t notice. Had they noticed,

it would not have mattered. They would not turn to misology. They saw the theorem well enough and already knew that the principles that it rested on needed further attention. They don't notice, but Plato does.

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# TENNENBAUM'S PROOF OF THE IRRATIONALITY OF $\sqrt{2}$

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The proof of the irrationality of  $\sqrt{2}$  involves proving that there cannot be positive integers  $n$  and  $m$  such that  $n^2 = 2m^2$ . This can be proved with a simple number-theoretic argument: First we note that  $n$  must be even, whence  $m$  must also be even, and hence both are divisible by 2. Then we observe that this is a contradiction if we assume that  $n$  is chosen minimally. There is also a geometric proof known already to Euclid, but the proof given by Tennenbaum seems to be entirely new. It is as follows: In Picture 1 we have on the left hand side two squares superimposed, one solid and one dashed. Let us assume that the area of the solid square is twice the area of the dashed square. Let us also assume that the side of each square is an integer and moreover the side of the solid square is as small an integer as possible. In the right hand side of Picture 1 we have added another copy of the dashed square to the lower left corner of the solid square, thereby giving rise to a new square in the middle and two small squares in the corners. The combined area of the two copies of the original dashed square is the same as the area of the original big solid square. In the superimposed picture the middle square gets covered by a dashed square twice while the small corner squares are not covered by the dashed squares at all. Hence the area of the middle square must equal the combined area of the two corner squares. But clearly all the squares have

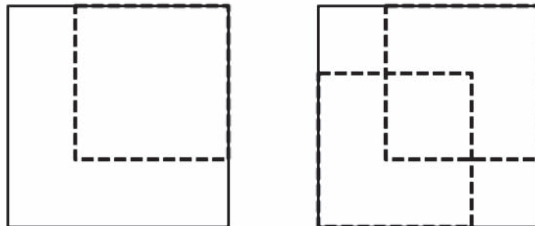


FIGURE 1. A proof of the irrationality of  $\sqrt{2}$ .

integer sides, so we get a contradiction with the minimality of the side of the solid square. If the picture is not convincing enough, this can be easily verified by a two-line calculation.

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